Density & Average of a Continuous Function

Density

Another application of integral is calculating total mass or population. A density function, normally denote $\rho(x)$, tells you how much of something there is at a point. The integral then adds up all of these points.

For example, a density function for population around a city with variable r would tell you how many people live at a point on the ring of the radius r from the city. You have to multiply by the circumference $2\pi r$ the get the population of the ring. The integral $\int_a^b \rho(r)\cdot 2\pi r\,dr$ would then add up all of the rings of radius a to b giving you a total population.

Examples:

fancy way to say along the rod

1. Find the total mass of a 5 meter rod whose linear density is given by $\rho(x)=\frac{e^x}{(1+e^x)^2}$ g/m for $0\leq x\leq 5$

$$\text{mass}=\int_0^5 \rho(x)\,dx=\int_0^5 \frac{e^x}{(1+e^x)^2}\,dx$$

$u=(1+e^x)$ $\qquad a'=1+e^0=2$

$du=e^x\,dx$ $\qquad b'=1+e^5$

$$\int_2^{1+e^5}\frac{1}{u^2}\,du$$
$$=\int_2^{1+e^5}u^{-2}\,du$$
$$=-1u^{-1}\Big|_2^{1+e^5}$$
$$=\left(-\frac{1}{1+e^5}\right)-\left(-\frac{1}{2}\right)$$
$$=\frac{1}{2}-\frac{1}{1+e^5}$$

2. A variety of deep sea worm is distributed about a hydrothermal vent according to the population density $\rho(r)=\frac{800}{9+r^2}$ thousand per square miles where $1\leq r\leq 3$ is the distance (in miles) from the tent. Find the total population of the sea worm.

Total worms on the circle of radius r $=\rho(r)\cdot$ perimeter of the circle

$=\rho\cdot(r)\cdot$ circumference of circle of radius r.

Total population $=\int_1^3$ (worms in circle of radius r) dr

$$=\int_1^3 \rho(r)\cdot 2\pi r\,dr$$
$$=\int_1^3 2\pi r\left(\frac{800}{9+r^2}\right)dr$$

$u=9+r^2 \quad du=2r\,dr$

$$=\int_{10}^{18} 800\pi\left(\frac{1}{u}\right)du$$
$$=800\pi\ln|u|\,du$$
$$=800\pi(\ln|18|-\ln|10|)$$
$$=800\pi\cdot\ln\left|\frac{9}{5}\right|$$

Average of a Function

The average of a continuous function $f(x)$ over the interval $[a,b]$ is given by,

$$f_{avg} = \frac{1}{b-a} \underbrace{\int_a^b f(x)\,dx}_{\text{total change}}.$$

Example:

1. Find the average amount of money over the first 10 years in an account earning interest at an annual rate of 4% compounded continuously if the principle is \$5000. Draw a graph of the balance in the account and mark the value that represents the average amount of money. Find the time it takes the account to reach this average.

Step 1: Set up average

$$f_{avg} = \frac{1}{b-a} \int_a^b A(t)\,dt$$

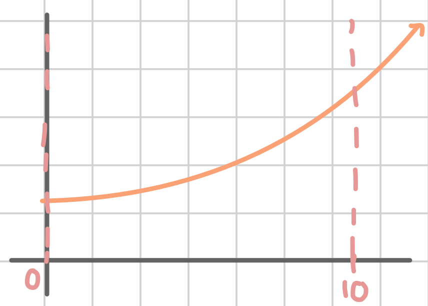
$A(t) = p_0 e^{rt}$

$p_0 = 5000$

$r = 0.04$

$$f_{avg} = \frac{1}{10} \int_0^{10} 5000 e^{0.04t}\,dt$$

Step 2: Draw a graph



Step 3: Solve average

$$f_{avg} = \frac{1}{10} \int_0^{10} 5000\, e^{0.04t}\,dt$$

$$= 500 \int_0^{10} e^{0.04t}\,dt$$

$$= 500 \cdot \left[e^{0.04t} \cdot \frac{1}{0.04} \right]_0^{10}$$

$$= 12500 \left[e^{0.04t} \right]_0^{10}$$

$$= 12500 \left(e^{0.04(10)} - e^{0.04(0)} \right)$$

$$= 12500 \left(e^{0.4} - 1 \right)$$

Step 4: Solve for T such that $A(T) = f_{avg}$

$$5000\, e^{0.04T} = 12500 \left(e^{0.4} - 1 \right)$$

$$e^{0.04T} = 2.5 \left(e^{0.04} - 1 \right)$$

$$\ln\left(e^{0.04T} \right) = \ln\left(2.5 \left(e^{0.04} - 1 \right) \right)$$

$$0.04T = \ln\left(2.5 \left(e^{0.04} - 1 \right) \right)$$

$$T = \frac{1}{0.04} \cdot \ln\left(2.5 \left(e^{0.04} - 1 \right) \right)$$

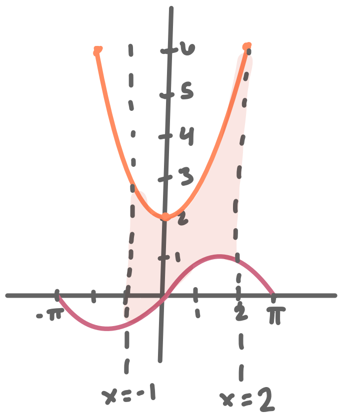
Exit Ticket Area Between Curves

Area Between curves Assuming that $f(x) \geq g(x)$ for $a \leq x \leq b$, the area between the curves is:

$$\int_a^b [f(x) - g(x)]\, dx$$

Set up but do NOT solve the integral that finds the areas bounded by the functions below:

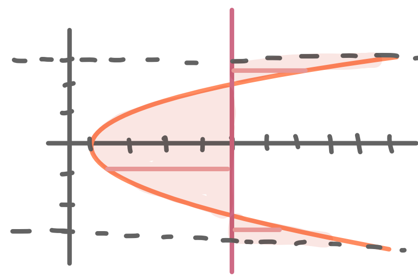
1. $y = x^2 + 2,\ y = \sin(x),\ x = -1,\ x = 2$



$\int_{-1}^{2} (x^2+2) - (\sin(x))\, dx$

2. $x = y^2 + 1,\ x = 5,\ y = -3,\ y = 3$

$y = \pm 2$ $x = 10$ $x = 10$

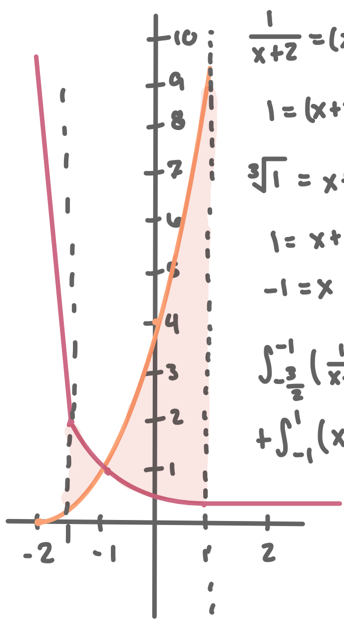


$\int_{2}^{3} (y^2+1) - (5)\, dy$

$\int_{-2}^{2} (5) - (y^2+1)\, dy$

$\int_{-3}^{-2} (y^2+1) - (5)\, dy$

3. $y = \frac{1}{x+2},\ y = (x+2)^2,\ x = -\frac{3}{2},\ x = 1$



$\frac{1}{x+2} = (x+2)^2$

$1 = (x+2)^3$

$\sqrt[3]{1} = x+2$

$1 = x+2$

$-1 = x$

$\int_{-\frac{3}{2}}^{-1} \left(\frac{1}{x+2}\right) - (x+2)^2\, dx$

$+ \int_{-1}^{1} (x+2)^2 - \left(\frac{1}{x+2}\right) dx$

4. $x = y^2 - y - 6,\ x = 2y + 4$

$x = (y-3)(y+2)$

intercepts:

$y^2 - y - 6 = 2y + 4$

$y - 3y - 10 = 0$

$(y-5)(y+2) = 0$

$y = 2, 5$

$\int_{-2}^{5} (2y+4) - (y^2 - y - 6)\, dy$