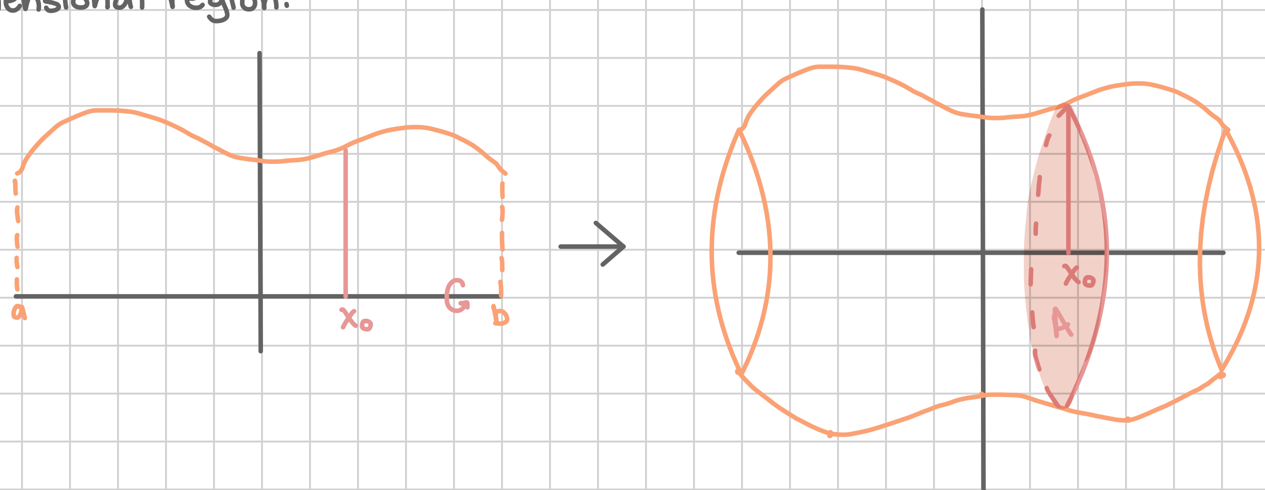
Volumes of Revolution

Volumes of Revolution

In this section we will start looking at the volume of a solid of revolution. To get a solid of revolution we start out with a function, $y = f(x)$ on an interval $[a,b]$. We then rotate this curve about a given axis to get the surface of the surface of the solid of revolution. For purposes of this discussion let's rotate the curve about the x-axis, although it could be any vertical or horizontal axis. Doing this for the below gives the following three dimensional region.



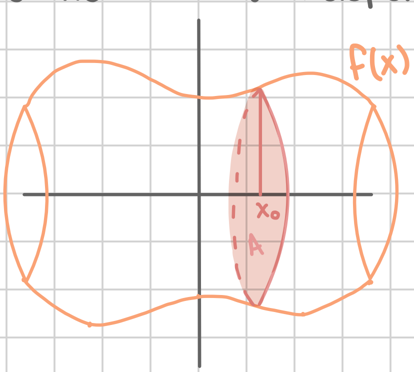
The volume formulas are $V = \int_a^b A(x)\,dx$ and $V = \int_c^d A(y)\,dy$ where $A(x)$ and $A(y)$ are the cross-sectional area functions of the solid. There are many ways to get the cross-sectional area. One of the easier methods for finding the cross-sectional area is to cut the object perpendicular to the axis of rotation. Doing this the cross-section will be either a solid disk if the object is solid (as the example above) or a washer if the solid has a hollowed out interior.

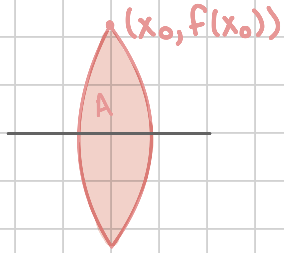
Disk Method

In the case that we get a solid disk the area is,

$$A = \pi\,(\text{radius})^2$$

where the radius will depend upon the function and the axis of rotation.





1-slice:
$A_i = 2\pi r$
$\quad = 2\pi f(x_i)$

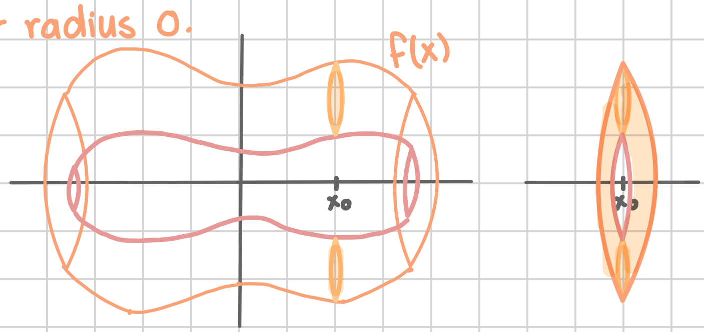
whole solid:
$V = \sum A_i \Delta x$
$\quad = \int_a^b A_i\,dx$

Washer Method

In the case that we get we get a washer the area is,

$$A = \pi((\text{outer radius})^2 - (\text{inner radius})^2)$$

where both of the radii will depend on the functions given and the axis of rotation. Note that these are the same concept as the solid disk just has inner radius 0.



1-slice: R = big radius, r = little radius

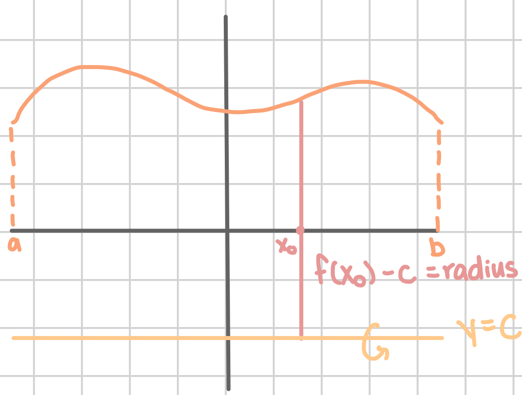
$$A_i = 2\pi R^2 - 2\pi r^2$$
$$= 2\pi[f(x)]^2 - 2\pi[g(x)]^2$$
$$= 2\pi([f(x)]^2 - [g(x)]^2)$$

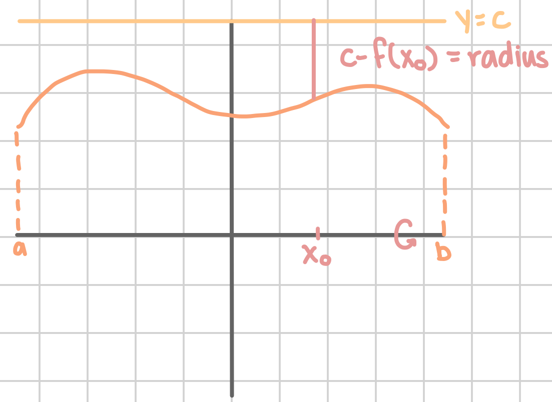
whole solid:

$$V = \sum A_i\,dx$$
$$= \int_a^b A_i\,dx$$

Axis of Rotation

The radius is highly dependent on the axis of rotation. Instead of rotating about the x-axis ($y=0$), we can rotate about a line $y=c$ and utilize the volume formula $V = \int_a^b \pi[f(x)-c]^2$ for disk method.



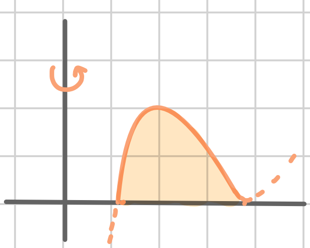


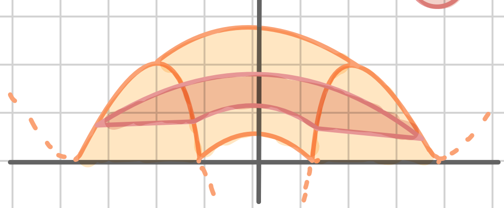
Shell Method

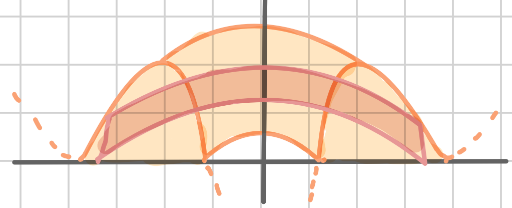
Sometimes we come across a solid that has the same inner and outer radius or they swap spots. In these cases we use a new method called shell method. Insted of adding up disks (or washers) to make a solid, it adds up cylinders. The volume formula is $V = \int_a^b 2\pi r h\,dx$

slice like a bagel

use a circular cookie cutter



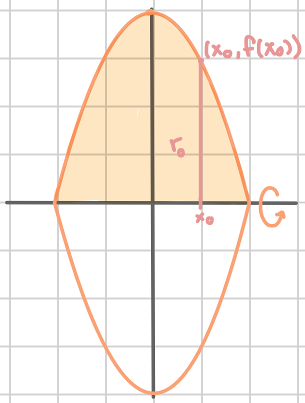


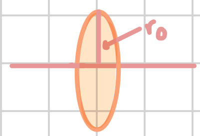


Examples:

1. Find the volume of the solid formed by rotating the region between the curve $y = 4 - x^2$ and x-axis for $-2 \le x \le 2$ about (a) the x-axis, (b) the line $y = -1$, and (c) the line $x = 3$.

(a) the x-axis

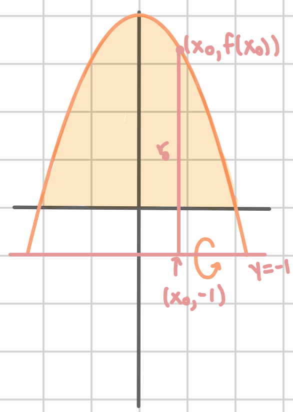


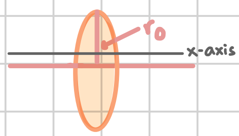


$A(x) = \pi(\text{radius})^2$
$= \pi(f(x))^2$
$= \pi(4-x^2)^2$
$= \pi(16 - 8x^2 + x^4)$

$V = \int_{-2}^{2} \pi(16 - 8x^2 + x^4)\,dx$
$= \pi \int_{-2}^{2} 16 - 8x^2 + x^4 \,dx$
$= \pi\left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_{-2}^{2}$
$= \pi\left[16(2) - \frac{8}{3}(2)^3 + \frac{1}{5}(2)^5 - \left(16(-2) - \frac{8}{3}(-2)^3 + \frac{1}{5}(-2)^5\right)\right]$
$= \pi\left[64 - \frac{128}{3} + \frac{64}{5}\right]$
$= \pi\left[\frac{960}{5} - \frac{640}{3} + \frac{192}{3}\right]$
$= \pi \frac{512}{15}$

(b) the line $y = -1$



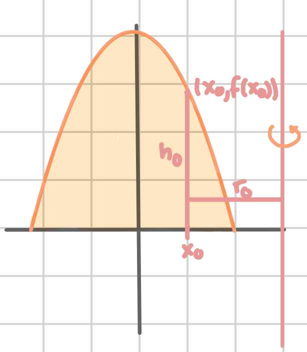


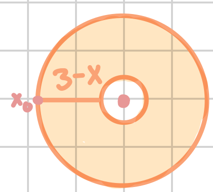
$r_0 = f(x_0) - (-1)$
$= f(x_0) + 1$

$A(x) = \pi(\text{radius})^2$
$= \pi((4-x^2) - (-1))^2$
$= \pi(5 - x^2)^2$

$V = \int_{-2}^{2} \pi(5 - x^2)^2\,dx$
$= \int_{-2}^{2} \pi(25 - 2x^2 + x^4)\,dx$
$= \pi \int_{-2}^{2} (25 - 2x^2 + x^4)\,dx$
$= \pi\left[25x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_{-2}^{2}$
$= \pi\left[25(2) - \frac{2}{3}(2)^3 + \frac{1}{5}(2)^5 - \left(25(-2) - \frac{2}{3}(-2)^3 + \frac{1}{5}(-2)^5\right)\right]$
$= \pi\left[100 - \frac{32}{3} + \frac{64}{5}\right]$
$= \pi\left[\frac{1500}{15} - \frac{160}{15} + \frac{192}{15}\right]$
$= \pi \frac{1532}{15}$

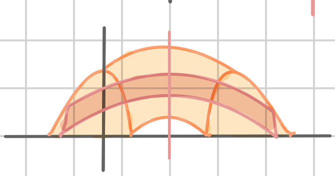
(c) the line $x = 3$





$A_0 = 2\pi r_0 h_0$
$= 2\pi(3 - x_0) \cdot f(x_0)$
$= 2\pi(3 - x_0)(4 - x_0^2)$

$V = \int_{-2}^{2} 2\pi(3-x)(4-x^2)\,dx$
$= 2\pi \int_{-2}^{2} (12 - 3x^2 - 4x + x^3)\,dx$
$= 2\pi\left[12x - x^3 - 2x^2 + \frac{1}{4}x^4\right]_{-2}^{2}$
$= 2\pi\left[12(2) - (2)^3 - 2(2)^2 + \frac{1}{4}(2)^4 - \left(12(-2) - (-2)^3 - 2(-2)^2 + \frac{1}{4}(-2)^4\right)\right]$
$= 2\pi[48 - 16 - 16]$
$= 2\pi[16]$
$= 32\pi$

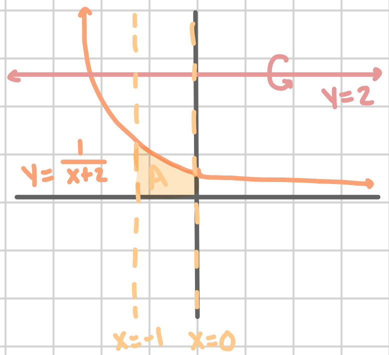


2. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

(a) $y = \frac{1}{x+2}$, $x = -1$, $x = 0$, $y = 0$; about the line $y = 2$.

(b) $y = -x^2 + 2x$, $y = x$; about the y-axis

(a)



method 1

$V =$ volume of rotating $y=0$ − volume of rotating $y = \frac{1}{x+2}$

$= \int_{-1}^{0} \pi (0-2)^2 \, dx - \int_{-1}^{0} \pi \left(\frac{1}{x+2} - 2\right)^2 dx$

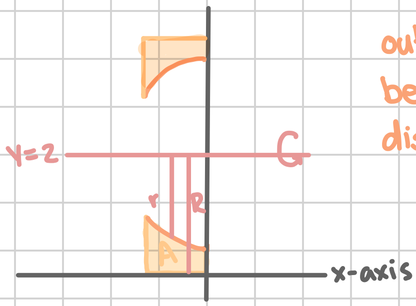
method 2

$V = \int_a^b \pi (\text{outer} - \text{inner})^2 dx$

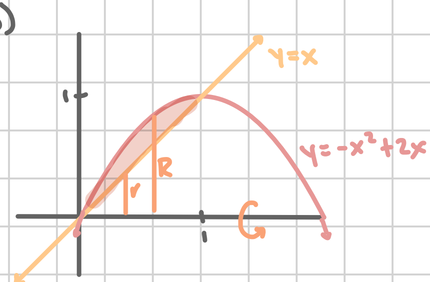
$= \int_{-1}^{0} \pi \left(2^2 - \left(2 - \frac{1}{x+2}\right)^2\right) dx$

outer radius is 2 because $y=0$ is distance 2 from $y=2$

similarly we have little r has distance 2 - the function



(b)



$V = \int_0^1 \pi (R^2 - r^2) dx$

$= \int_0^1 \pi \left((-x^2+2x)^2 - (x)^2\right) dx$

$= \int_0^1 \pi (x^4 - 4x^3 + 4x^2 - x^2) \, dx$

$= \pi \int_0^1 x^4 - 4x^3 + 3x^2 \, dx$

$= \pi \left[\frac{1}{5}x^5 - x^4 + x^3 \right]_0^1$

$= \pi \left(\frac{1}{5} - 1 + 1 \right)$

$= \frac{1}{5}\pi$

Exit Ticket Volume of Solids with Uniform Cross-sections

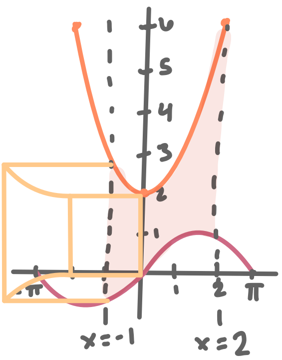
Volume of Solids with Uniform Cross-sections Consider a solid whose base is the region bounded by given function(s) with uniform cross-sections perpendicular to the x-axis. The volume of the solid is given by:

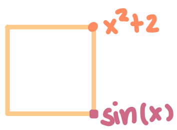
$$V = \int_a^b [A(x)]\, dx$$

where $A(x)$ is the area of the cross-section

Set up but do NOT solve the integral that finds the volume of the solid whose base is bounded by $y = x^2 + 2,\ y = \sin(x),\ x = -1,\ x = 2$ and has uniform cross-sections perpendicular to the x-axis in the shape of:

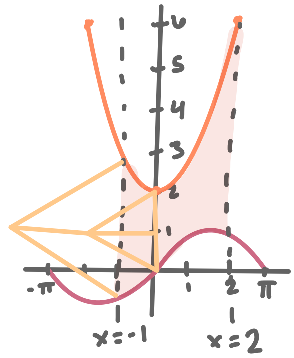
1. squares

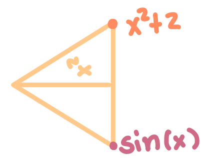




$$A(x) = (\text{base})^2$$
$$= ((x^2+2)-(\sin(x)))^2$$
$$V = \int_{-1}^{2} (x^2+2-\sin(x))^2\, dx$$

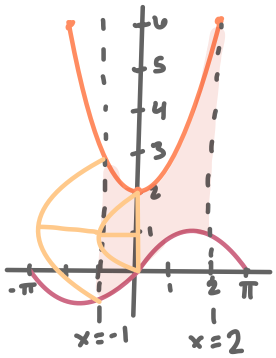
2. triangles of height x^2

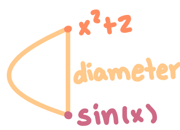




$$A(x) = \tfrac{1}{2}(\text{base})(\text{height})$$
$$= \tfrac{1}{2}(x^2+2-\sin(x))(x^2)$$
$$V = \int_{-1}^{2} \tfrac{1}{2}x^4 + x^2 - \tfrac{1}{2}x^2\sin(x)\, dx$$

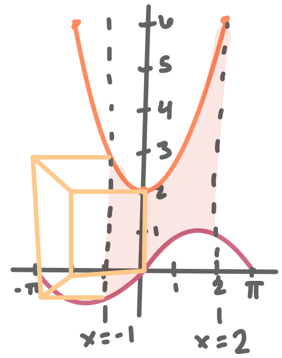
3. semicircles

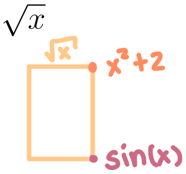




$$A(x) = \tfrac{1}{2}\pi r^2$$
$$= \tfrac{1}{2}\pi\left(\tfrac{1}{2}d\right)^2$$
$$= \tfrac{1}{8}\pi d^2$$
$$= \tfrac{1}{8}\pi(x^2+2-\sin(x))$$
$$V = \int_{-1}^{2} \tfrac{1}{8}\pi(x^2+2-\sin(x))\, dx$$

4. rectangles of height $\sqrt{x}$





$$A(x) = (\text{base})(\text{height})$$
$$= (x^2+2-\sin(x))(\sqrt{x})$$
$$V = \int_{-1}^{2} x^{5/2} + 2\sqrt{x} - \sqrt{x}\sin(x)\, dx$$