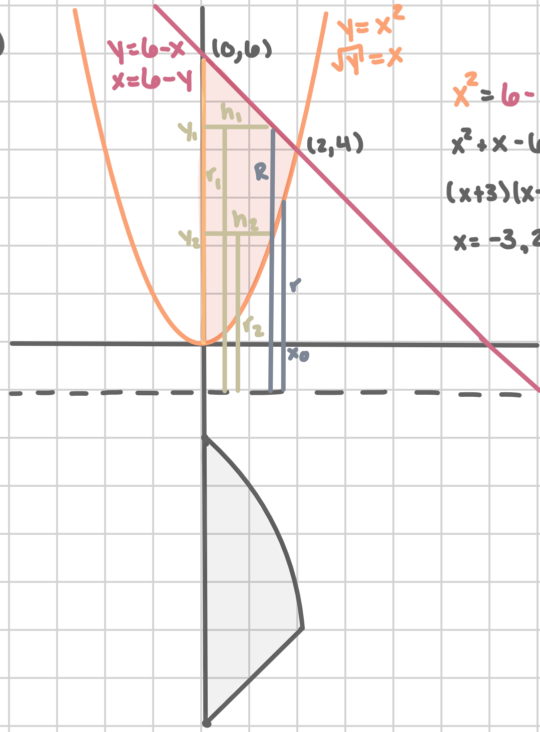
Volumes of Revolution

Examples:

1. Find the volume of the solid obtained by rotating the region in the first quad. bounded by the given curves about the specified line. Your construction should have one single integral for the volume you like to find.
   (a) $y=x^2$, $y=6-x$, $x=0$ about the line $y=-1$
   (b) $y=x^2$, $y=6-x$, $y=0$ about the line $y=-1$

(a)



$x^2=6-x$

$x^2+x-6=0$

$(x+3)(x-2)=0$

$x=-3,2$

option 1. washer method

only one R,r option = one integral

$$\int_0^2 \pi[\underbrace{((6-x)-(-1))}_{R=\text{top - bottom}}{}^2 - \underbrace{((x^2)-(-1))}_{r=\text{top - bottom}}{}^2]dx$$

"bottom" is shifted axis

"bottom" is shifted axis

option 2. shell method

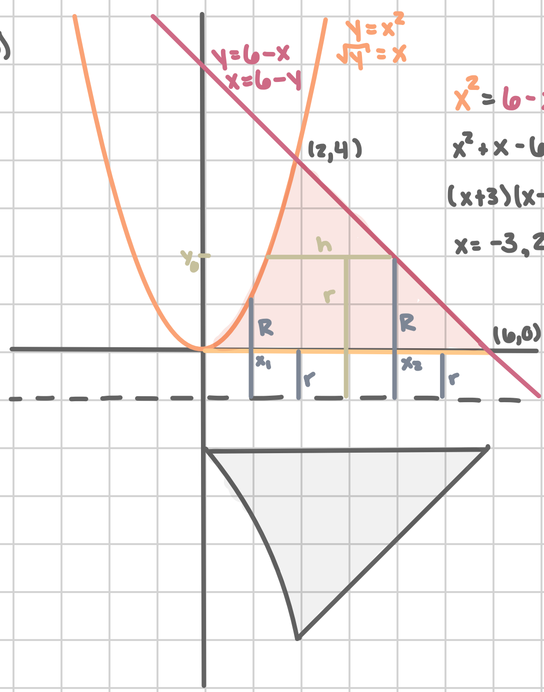
height has two options = two integrals

$$\int_0^4 2\pi\underbrace{(y-(-1))}_{r=\text{top-bottom}}(\sqrt{y})\,dy + \int_4^6 2\pi\underbrace{(y-(-1))}_{r=\text{top-bottom}}(6-y)\,dy$$

"bottom" is shifted axis

"bottom" is shifted axis

(b)



$x^2=6-x$

$x^2+x-6=0$

$(x+3)(x-2)=0$

$x=-3,2$

option 1. washer method

big R has two options = two integrals

$$\int_0^2 \pi((x^2+1)^2-(1)^2)\,dx + \int_2^6 \pi((6-x+1)^2-(1)^2)\,dx$$

shifted axis of rotation

$r=(x^2)-(-1)$

shifted axis of rotation

$r=(6-x)-(-1)$

option 2. shell method

only one height and radius = one integral

$$\int_0^4 2\pi(y+1)\underbrace{((6-y)-(\sqrt{y}))}_{h=\text{right - left}}\,dy$$

shifted axis of rotation

slice depends on y not x

Work and Energy

Work and Energy

In this section we will be looking at the amount of work that is done by a force in moving an object. In physics you learn that work's formula is $W = F \cdot d$ where F is a constant force and d is the distance traveled over. However, force is not always constant.

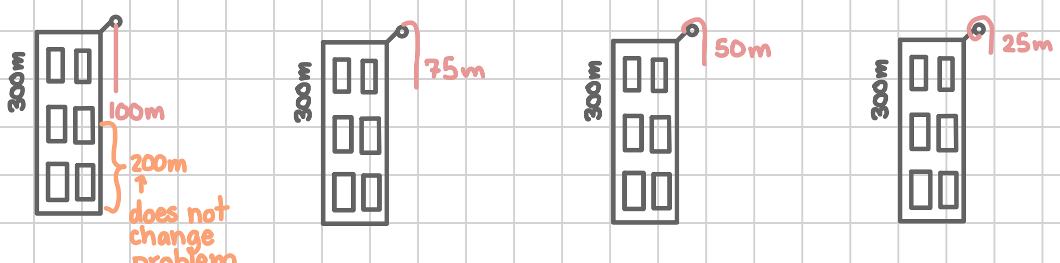
Suppose that the force at any x is given by $F(x)$ then the work done by the force in moving the object from $x=a$ to $x=b$ is given by $W = \int_a^b F(x)\,dx$.
Notice that if $F(x)$ is a constant then $\int_a^b F\,dx = F\cdot x\,\Big]_a^b = F(b-a) = F\cdot d$

Examples:

2. A 300 kg chain 100 m in length is attached at one end to a crank on the top of a 300 m building, and the rest of the chain is allowed to hang freely on the side of the building from the crank.
   (a) How much work is done when the whole chain is cranked up to the top of the tower?
   (b) If a 20 kg weight is attached to the bottom end of the chain, how would the amount of work change?
   (c) If a 20 kg weight is attached to the bottom end of the chain, how much work is done to crank 3/4 of the chain with the weight attached to the top of the tower?

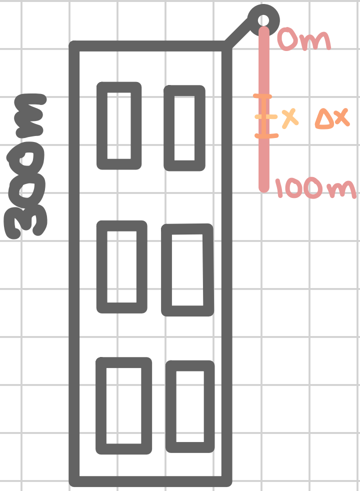
(a) Let's make a movie of what is happening.

initial: quarter done: half way: almost there:



As we lift the chain less and less hangs off the side making the weight we are moving less and less as time goes on. This gives the variable force need to lift the chain. So how much would it take to move only a small piece?

Let's label the graph with values to help.



How much work does it take to move the tiny sliver around x up to the top?

W_x = force · displacement   we assume constant at x

= (mass · gravity) · displacement   force = mass · acceleration

= (density · length) · gravity · displacement   300kg = total mass

density = $\frac{\text{total mass}}{\text{total length}}$ tiny piece use Δx to move x to the top x moves x length

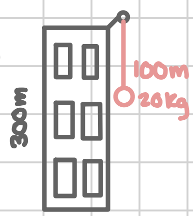
$W_x = \left(\frac{300}{100}\right)(\Delta x)(10) \cdot x$

Now add up all of the tiny slivers:

x domain Δx becomes dx

$$W_T = \int_0^{100} 30x \, dx$$
$$= 15x^2 \Big|_0^{100}$$
$$= 150{,}000$$

(b)



The ball does not change weight therefore the force is not variable and we can use W = force · displacement.

W = $W_{chain} + W_{ball}$

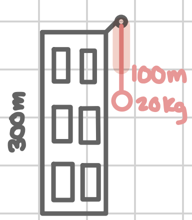
= part a + (mass · gravity) · (displacement)

= 150,000 + (20 kg)(10 m/s²)(100 m)

= 150,000 + 20,000

= 170,000

(c)



Now we only want to move the chain and ball 75m. How does the work change in each part?

Note that the first $\frac{3}{4}$ of the chain has variable force, but the last $\frac{1}{4}$ still requires work to move 75m.

$W = W_{75} + W_{25} + W_{ball}$

$$= \int_0^{75} 30x \, dx + \underbrace{(3)(25)}_{\text{mass}}\underbrace{(10)}_{g}\underbrace{(75)}_{\text{displacement}} + (20)(10)\underbrace{(75)}_{\text{displacement}}$$