Integration by Parts

Integration Techniques

Let's start with some integrals we already know how to solve:
First, we see an integral that we know the formula for,

$$\int e^x dx = e^x + c.$$

Next, we consider something more complicated with a u and a u',

$$\int x e^{x^2} dx = \int u' e^u dx$$

$u = x^2$

$du = 2x dx \Rightarrow \frac{1}{2} du = x dx$

$$= \frac{1}{2} \int e^u du$$
$$= \frac{1}{2} e^u + c$$
$$= \frac{1}{2} e^{x^2} + c$$

These are problems we expect you to recognize and be able to solve with relative ease and speed.

Now we look at a new type of integral, $\int x e^{6x} dx$.

If this contained only the x or only the e^{6x} then we could solve the integral. If the integrand was $x e^{6x^2}$ then we also could solve it using u-substitution. However, neither is the case and we do not have the knowledge to solve this.

We introduced a new process called integration by parts. This method is the undoing of the product rule.

product rule:	$(fg)' = f'g + g'f$
integrate both sides:	$\int (fg)' dx = \int f'g \, dx + \int g'f \, dx$
solve for $\int g'f \, dx$:	$\int g'f \, dx = \int (fg)' dx - \int f'g \, dx$
simplify:	$\int f \cdot g' dx = f \cdot g - \int f'g \, dx$
substitution:	$\int u \, dv = uv - \int v \, du$

I write this formula
$$\int u \cdot dv = uv - \int v \, du$$
where $u = f(x)$, $v = g(x)$,
$du = f'(x) dx$, $dv = g'(x) dx$
and use the mnemonic device "ultra violet voodoo"

The challenging part of this method is deciding what should be u and what should be dv It is not always clear and sometime we will make the wrong choice and have to start over. We know we have made the right choice when we can fill out the formula and $\int v \, du$ is something we can integrate.
I use the acronym LIATE (log, inverse trig, algebra, trig, exponential) to pick u.

Product Rule for Integrals: $\int u \, dv = uv - \int v \, du$

Examples:

1. Evaluate the integral $\int xe^{6x}dx$.

$\int xe^{6x}\,dx$

$u=x \qquad dv=e^{6x}dx$

$du=dx \qquad v=\int e^{6x}dx=\frac{1}{6}e^{6x}$

$=x\cdot\frac{1}{6}e^{6x}-\int\frac{1}{6}e^{6x}\cdot dx$

$u=6x$

$du=6dx \Rightarrow \frac{1}{6}du=dx$

$=\frac{1}{6}xe^{6x}-\frac{1}{6}\int e^{u}\cdot\frac{1}{6}du$

$=\frac{1}{6}xe^{6x}-\frac{1}{36}e^{u}+c$

$=\frac{1}{6}xe^{6x}-\frac{1}{36}e^{6x}+c$

notice that picking $u=x$ means the x drops out and the resulting integral is something we can do. picking $u=e^{6x}$ would just result in the integral getting worse when we aim for it to go away

2. Evaluate the follow integrals using integration by parts.

(a) $\int x^3\ln(x)\,dx$

$u=\ln(x) \qquad dv=x^3dx$

$du=\frac{1}{x}dx \qquad v=\frac{1}{4}x^4$

$=\ln(x)\cdot\frac{1}{4}x^4-\int\frac{1}{4}x^4\cdot\frac{1}{x}\,dx$

$=\frac{1}{4}x^4\cdot\ln(x)-\frac{1}{4}\int x^3dx$

$=\frac{1}{4}x^4\cdot\ln(x)-\frac{1}{4}\left(\frac{1}{4}x^4\right)+c$

(b) $\int\arctan(x)\,dx$

$u=\arctan(x) \qquad dv=1dx$ ← there is always a 1 hidden

$du=\frac{1}{1+x^2}dx \qquad v=x$

$=x\arctan(x)-\int x\cdot\frac{1}{1+x^2}\,dx$

$u=1+x^2$

$du=2x\,dx$

$=x\arctan(x)-\frac{1}{2}\ln(1+x^2)+c$

(c) $\int x\cos(3x+2)\,dx$ ← feels like u-sub but not exactly du

↪ choosing $u=\cos(3x+2)$ would not "fix" the product

$u=x \qquad dv=\cos(3x+2)$

$du=1dx \qquad v=\frac{1}{3}\sin(3x+2)$

$=x\cdot\frac{1}{3}\sin(3x+2)-\int\frac{1}{3}\sin(3x+2)\cdot 1dx$

$u=3x+2$

$du=3dx$

$=\frac{1}{3}x\sin(3x+2)-\frac{1}{3}\left(-\frac{1}{3}\cos(3x+2)\right)+c$

if $u=\sin(x) \qquad dv=x$

$du=\cos(x)dx \qquad v=\frac{1}{2}x^2$

the $=\sin(x)\cdot\frac{1}{2}x^2-\underbrace{\int\frac{1}{2}x^2\cos(x)dx}_{\text{more complex}}$

Integration Techniques:

1. Is it a rule I Know?
   e.g. $\int\sin(x)dx$, $\int\frac{1}{\sqrt{1-x^2}}dx$, $\int e^{2x}dx$, etc.

2. Is there a function & its derivative?
   e.g. $\int x\cdot\sin(x^2)dx$, $\int\frac{2x}{\sqrt{1-x^2}}dx$, $\int xe^{x^2}dx$, etc.
   ↪ u-substitution $\int f(g(x))\cdot g'(x)dx=\int f(u)du$

3. Is it a product that seems unrelated?
   $\int x\cdot\sin(x)dx$, $\int e^{x}\cdot\ln(x)dx$, $\int\arcsin(x)\cdot 1dx$, etc.
   ↪ integration by parts

Another integration technique that (rarely) arises from integration by parts:

(a) $\int x^2 \cos(x)\,dx$

$u = x^2$ $\quad dv = \cos(x)\,dx$

$du = 2x\,dx$ $\quad v = \sin(x)$

$= x^2 \cdot \sin(x) - \underbrace{\int \sin(x) \cdot 2x\,dx}$

simpler but still integration by parts

$u = 2x$ $\quad dv = \sin(x)\,dx$

$du = 2dx$ $\quad v = -\cos(x)$

careful of negative

$= x^2 \cdot \sin(x) - [x \cdot -\cos(x) - \int -\cos(x) \cdot 2\,dx]$

$= x^2 \cdot \sin(x) + x\cos(x) - 2\int \cos(x)\,dx$

$= x^2 \cdot \sin(x) + x\cos(x) - 2\sin(x) + C$

(b) $\int e^x \cos(x)\,dx$

$u = \cos(x)$ $\quad dv = e^x dx$

$du = -\sin(x)\,dx$ $\quad v = e^x dx$

$= \cos(x) \cdot e^x - \int e^x \cdot -\sin(x)\,dx$

$u = -\sin(x)dx$ $\quad dv = e^x dx$

$du = -\cos(x)dx$ $\quad v = e^x$

$= e^x \cos(x) - [-\sin(x) \cdot e^x - \int e^x \cdot -\cos(x)\,dx]$

$= e^x \cos(x) + e^x \cdot \sin(x) - \underbrace{\int e^x \cdot \cos(x)\,dx}$

this is the original integral
we can add it back to the other side to get 2·integral

$2\int e^x \cos(x)\,dx = e^x \cdot \cos(x) + e^x \cdot \sin(x)$

$\int e^x \cos(x)\,dx = \frac{1}{2}(e^x \cdot \cos(x) + e^x \cdot \sin(x))$

Exit Ticket Work and Energy

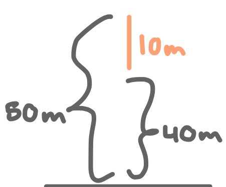
> **Work and Energy** Suppose that the force at any given x is given by $F(x)$, then the work done by the force in moving the object from $x = a$ to $x = b$ is given by
>
> $$W = \int_a^b F(x)dx.$$

Set up but do NOT solve the following integral:

1. A uniform chain 10 m long weighing 30 kg lying completely at the foot of a building 50 m tall.

   (a) What is the work done against gravity to move one end to the top of the building with the rest of the chain danging free?

   bottom of chain does not move 50m only 40



   W_y = force · displacement

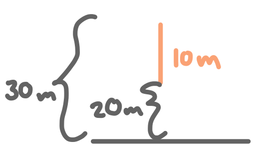
   = density · length · gravity · displacement

   $= \left(\frac{30}{10}\right) \cdot (\Delta y) \cdot (10) \cdot (50-y)$

   $W = \int_0^{10} 30(50-y)\,dy$

   (b) What is the work done to move one end only 30 m off the ground?

   top only moves 30m off and the bottom moves 20m



   W_y = density · length · gravity · displacement

   $= \left(\frac{30}{10}\right) \cdot (\Delta y) \cdot (10) \cdot (20-y)$

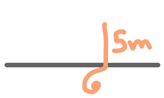
   $W = \int_0^{10} 30(20-y)\,dy$

   (c) What is the work done to move the top end of the chain 5 meters off the ground with the rest of the chain still on the ground?

   the bottom of the chain doesn't move ⇒ adds no work

   W_y = density · length · gravity · displacement

   $= \left(\frac{30}{10}\right) \cdot (\Delta y) \cdot (10) \cdot (5-y)$



   $W = \int_0^{5} 30(5-y)\,dy$