# Trigonometric Integrals

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Let us start with an integral that we know how to do,

$$\int \cos(x)\cdot\sin^5(x)\,dx$$

$u=\sin(x)$

$du=\cos(x)\,dx$

$$=\int u^5\,du$$
$$=\frac{1}{6}u^6+c$$
$$=\frac{1}{6}(\sin(x))^6+c$$

This integral is easy to do with a substitution because the presence of the cosine
Let us consider it without,

$$\int \sin^5(x)\,dx$$

Notice that we are unable to do the u-substitution without the cosine, so we may try to reintroduce it using identities,

$$\int \sin^5(x)\,dx$$
$$=\int \sin^4(x)\cdot\sin(x)\,dx$$
$$=\int (\sin^2(x))^2\cdot\sin(x)\,dx \qquad \text{utilize } \sin^2(x)+\cos^2(x)=1 \Rightarrow \sin^2(x)=1-\cos^2(x)$$
$$=\int (1-\cos^2(x))^2\cdot\sin(x)\,dx$$

Now that we have both sine and cosine, we can reintroduce the u-substition

$$\int (1-\cos^2(x))^2\cdot\sin(x)\,dx$$

$u=\cos(x)$

$du=-\sin(x)\,dx$

$$=-\int (1-u^2)^2\cdot du$$
$$=-\int 1-2u^2+u^4\,du$$
$$=-\left[u-\frac{2}{3}u^3+\frac{1}{5}u^5\right]+c$$
$$=-\cos(x)+\frac{2}{3}(\cos(x))^3-\frac{1}{5}(\cos(x))^5+c$$

Notice that this rewriting and substitution worked because the exponent was odd, one sine stays and the rest get changed. It is often good practice to separate the odd function so that we have one sine (or cosine) and the rest cosine (or sine).

Recap: $\int \sin^n(x)\cos^m(x)\,dx$

if n is odd: remove 1 sine, substitute the rest to cosine using $\sin^2(x)=1-\cos^2(x)$, use substitution $u=\cos(x)$

if m is odd: remove 1 cosine, substitute the rest to sine using $\cos^2(x)=1-\sin^2(x)$, use substitution $u=\sin(x)$

if n and m are odd: choose the one with the smallest exponent and follow that path

Example:

1. $\int \sin^6(x)\cos^3(x)\,dx$ ← cos(x) is odd
$= \int \sin^6(x)\cdot\cos^2(x)\cdot\cos(x)\,dx$ save one & replace the rest
$= \int \sin^6(x)\cdot(1-\sin^2(x))\cdot\cos(x)\,dx$
$u = \sin(x)$
$du = \cos(x)\,dx$
$= \int u^6(1-u^2)\cdot du$
$= \int u^6 - u^8\,du$
$= \frac{1}{7}u^7 - \frac{1}{9}u^9 + c$
$= \frac{1}{7}(\sin(x))^7 - \frac{1}{9}(\sin(x))^9 + c$

Now we ask ourselves, what if m and n are even?

2. $\int \sin^2(x)\cdot\cos^2(x)\,dx$
$= \int [\frac{1}{2}(1-\cos(2x))]\cdot[\frac{1}{2}\cdot(1+\cos(2x))]\,dx$ half-angle $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$
$= \frac{1}{4}\int 1-\cos^2(2x)\,dx$
$= \frac{1}{4}\int 1-[\frac{1}{2}(1+\cos(4x))]\,dx$ half angle $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$
$= \frac{1}{4}\int 1-\frac{1}{2}-\frac{1}{2}\cos(4x)\,dx$
$= \frac{1}{4}\int \frac{1}{2}-\frac{1}{2}\cos(4x)\,dx$
$= \frac{1}{4}[\frac{1}{2}x - \frac{1}{8}\sin(4x)] + c$
$= \frac{1}{8}x - \frac{1}{32}\sin(4x) + c$

alternatively. $\int \sin^2(x)\cdot\cos^2(x)\,dx$
$= \int (\sin(x)\cdot\cos(x))^2\,dx$ double angle $\sin\theta\cos\theta = \frac{1}{2}\sin(2\theta)$
$= \int (\frac{1}{2}\sin(2x))^2\,dx$
$= \frac{1}{4}\int \sin^2(2x)\,dx$
$= \frac{1}{4}\int \frac{1}{2}(1-\cos(4x))\,dx$ half angle $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$
$= \frac{1}{8}\int 1-\cos(4x)\,dx$
$= \frac{1}{8}[x - \frac{1}{4}\sin(4x)] + c$
$= \frac{1}{8}x - \frac{1}{32}\sin(4x) + c$

In both of these examples we have sine and cosine of the same angle, but what if they are different?

3. $\int \cos(15x)\cos(4x)\,dx$
$= \int \frac{1}{2}[\cos(15x-4x) + \cos(15x+4x)]\,dx$ $\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$
$= \frac{1}{2}\int \cos(11x) + \cos(19x)\,dx$
$= \frac{1}{2}[\frac{1}{11}\sin(11x) + \frac{1}{19}\sin(19x)] + c$

Now that we have covered all of the sine/cosine cases, we next consider the secant/tangent cases.

4. (a) $\int \sec^9(x)\tan^5(x)\,dx$

$= \int \sec^8(x)\tan^4(x)\cdot\tan(x)\sec(x)\,dx$

$= \int \sec^8(x)(\sec^2(x)-1)^2\cdot\tan(x)\sec(x)\,dx$

$u = \sec(x)$

$du = \sec(x)\tan(x)$

$= \int u^8(u^2-1)^2\,du$

$= \int u^{12} - 2u^{10} + u^8\,du$

$= \frac{1}{13}u^{13} - \frac{2}{11}u^{11} + \frac{1}{9}u^9 + C$

$= \frac{1}{13}(\sec(x))^{13} - \frac{2}{11}(\sec(x))^{11} + \frac{1}{9}(\sec(x))^9 + C$

we left 1 sin or cos earliear so that we had du for u=cos/sin. when $u=\sec(x)$ we need $du=\sec(x)\tan(x)$

trig rule: $1+\tan^2(x)=\sec^2(x)$

(b) $\int \tan^3(x)\,dx$

$= \int \tan(x)\cdot\tan^2(x)\,dx$

$= \int \tan(x)\cdot(\sec^2(x)-1)\,dx$

$= \int \tan(x)\cdot\sec^2(x)\,dx - \int \tan(x)\,dx$

$= \int \tan(x)\cdot\sec^2(x)\,dx - \int \frac{\sin(x)}{\cos(x)}\,dx$

$u = \tan(x)$ $\qquad v = \cos(x)$

$du = \sec^2(x)dx$ $\qquad dv = -\sin(x)\,dx$

$= \int u\,du \qquad - \int \frac{1}{v}\,dv$

$= \frac{1}{2}u^2 - \ln|v| + C$

$= \frac{1}{2}(\tan(x))^2 - \ln|\sec(x)| + C$

(c) $\int \sec(x)\,dx$

$= \int \frac{\sec(x)\cdot(\sec(x)+\tan(x))}{\sec(x)+\tan(x)}\,dx$

$= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx$

$u = \sec(x)+\tan(x)$

$du = \sec(x)\cdot\tan(x) + \sec^2(x)\,dx$

$= \int \frac{1}{u}\,du$

$= \ln|u| + C$

$= \ln|\sec(x)+\tan(x)| + C$

## List of Trigonometric Identities

$\cos^2\theta + \sin^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$\cot^2\theta + 1 = \csc^2\theta$

$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$

$\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$

$\sin\theta\cos\theta = \frac{1}{2}\sin(2\theta)$

$\sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$

$\cos A\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]$

$\sin A\sin B = -\frac{1}{2}[\cos(A+B)-\cos(A-B)]$

## **Exit Ticket** Integration by Parts

> **Integration by Parts** Let $u(x)$ and $v(x)$ be two differentiable functions. Integration by parts says
>
> $$\int udv = ux - \int vdu$$

Evaluate the following integrals:

**1.** $\int 8xe^{6x}dx$

$u = 8x \qquad dv = e^{6x}dx$

$du = 8dx \qquad v = \frac{1}{6}e^{6x}$

$= 8x \cdot \frac{1}{6}e^{6x} - \int \frac{1}{6}e^{6x} \cdot 8\,dx$

$= \frac{4}{3}xe^{6x} - \frac{2}{9}e^{6x} + C$

**2.** $\int 4x\cos(2-3x)dx$

$u = 4x \qquad dv = \cos(2-3x)dx$

$du = 4dx \qquad v = -\sin(2-3x) \cdot \frac{1}{3}$

$= -4x \cdot \sin(2-3x) \cdot \frac{1}{3} + \int \frac{1}{3}\sin(2-3x) \cdot 4dx$

$= -\frac{4}{3}x\sin(2-3x) + \frac{4}{3}\cos(2-3x) \cdot \frac{1}{3}$

**3.** $\int (2-x)^2 ln(4x)dx$

$u = \ln(4x) \qquad dv = (2-x)^2\,dx$

$du = \frac{1}{x} \qquad v = \frac{1}{3}(2-x)^3 \cdot -1$

$= -\ln(4x) \cdot \frac{1}{3}(2-x)^3 + \int \frac{1}{3}\frac{(2-x)^3}{x}\,dx$

$= -\frac{1}{3}\ln(4x)(2-x)^3 + \frac{1}{3}\int \frac{x^3 - 6x^2 - 12x}{x}\,dx$

$= -\frac{1}{3}\ln(4x)(2-x)^3 + \frac{1}{3}(\frac{1}{3}x^3 - 3x^2 - 12x) + C$

**4.** $\int \ln(x)dx$

$u = \ln(x) \qquad dv = 1dx$

$du = \frac{1}{x}dx \qquad v = x$

$= \ln(x) \cdot x - \int x \cdot \frac{1}{x}dx$

$= x \cdot \ln(x) - x + C$

**5.** $\int e^{-x}\sin(4x)dx$

$u = \sin(4x) \qquad dv = e^{-x}dx$

$du = 4\cos(4x)dx \qquad v = -e^{-x}$

$= -\sin(4x) \cdot e^{-x} + \int e^{-x} \cdot 4\cos(4x)\,dx$

$u = 4\cos(4x) \qquad dv = e^{-x}dx$

$du = -16\sin(4x)dx \qquad v = -e^{-x}$

$= -\sin(4x)e^{-x} - 4\cos(4x)e^{-x} - \int e^{-x} \cdot 16\sin(4x)\,dx$

$17\int e^{-x}\sin(4x)dx = -\sin(4x)e^{-x} - 4\cos(4x)e^{-x} + C$

$\int e^{-x}\sin(4x)dx = \frac{1}{17}[-\sin(4x)e^{-x} - 4\cos(4x)e^{-x} + C]$

**6.** $\int \frac{x^7}{\sqrt{x^4+1}}dx$

$u = x^4 \qquad dv = \frac{x^3}{\sqrt{x^4+1}}dx$

$du = 4x^3dx \qquad v = \frac{1}{2}(x^4+1)^{1/2}$

$= x^4 \cdot \frac{1}{2}(x^4+1)^{1/2} - \int \frac{1}{2}(x^4+1)^{1/2} \cdot 4x^3dx$

$u = x^4+1$

$du = 4x^3$

$= \frac{1}{2}x^4(x^4+1)^{1/2} - \int \frac{1}{2}u^{1/2}\,du$

$= \frac{1}{2}x^4(x^4+1)^{1/2} - \frac{1}{3}u^{3/2} + C$

$= \frac{1}{2}x^4(x^4+1)^{1/2} - \frac{1}{3}(x^4+1)^{3/2} + C$

$\int \frac{x^3}{\sqrt{x^4+1}}dx$

$u = x^4+1$

$du = 4x^3dx$

$\int u^{-1/2} \cdot \frac{1}{4}du$

$2 \cdot u^{1/2} \cdot \frac{1}{4} + C$

$\frac{1}{2}(x^4+1)^{1/2} + C$