Partial Fraction

Let's go back to algebra for a moment. How do you combine fractions with different denominators?

$$\frac{4}{x-3} - \frac{1}{x+2}$$

$$\frac{4(x+2)}{(x-3)(x+2)} - \frac{1(x-3)}{(x+2)(x-3)}$$ multiply by missing factor to get common denominator

$$\frac{4x+8}{(x-3)(x+2)} - \frac{x-3}{(x+2)(x-3)}$$

$$\frac{4x+8-x+3}{(x-3)(x+2)}$$

$$\frac{3x+11}{(x-3)(x+2)}$$

Like many in integral sections, we ask how can we go backwards?

i.e. $\frac{3x+11}{(x+2)(x-3)} = \frac{A}{(x+2)} + \frac{B}{(x-3)}$ where A and B are expressions that may contain x.

Partial Fraction Decomposition

The easiest way to learn this method is to see it done.

1. $\frac{5x-4}{x^2-x-2}$ first check that the numerator has lower degree than denominator, if not do long division

$$\frac{5x-4}{x^2-x-2} = \frac{5x-4}{(x-2)(x+1)}$$ factor the bottom

$$\frac{5x-4}{(x-2)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)}$$ write a partial fraction for each factor (see chart for all expansions)

$$\frac{5x-4}{(x-2)(x+1)} = \frac{A(x+1)}{(x-2)(x+1)} + \frac{B(x-2)}{(x-2)(x+1)}$$ multiple by common denominator so it disappears

$$5x-4 = A(x+1) + B(x-2)$$

method 1: roots

Plug in roots to recieve 0 terms

$x=-1$: $5(-1)-4 = A(-1+1) + B(-1-2)$

$-9 = A\cdot 0 + B\cdot -3$

$3 = B$

$x=2$: $5(2)-4 = A(2+1) + B(2-2)$

$6 = 3A + 0B$

$2 = A$

$$\frac{5x-4}{x^2-x-2} = \frac{2}{x-2} + \frac{3}{x+1}$$

Warning: this method only works when you have no repeated factors

method 2: linear algebra

multiple out

$5x-4 = Ax + A + Bx - 2B$

separate the powers

$5x = Ax + Bx$

$-4 = A - 2B$

solve system of linear equations

$5 = A + B \Rightarrow A = 5 - B \Rightarrow A = 5-3$

$\underbrace{-4 = A - 2B}$ $\quad -4 = (5-B) - 2B \quad A = 2$

solve for A in eq. 1 and plug into eq. 2

$-4 = 5 - 3B$

$-9 = -3B$

$3 = B$

$$\frac{5x-4}{x^2-x-2} = \frac{2}{x-2} + \frac{3}{x+1}$$

2. $\dfrac{x^2+15}{(x+3)^2(x^2+3)}$

$(x+3)^2$ has an exponent of 2 so it splits into two partial fraction $\frac{A}{(x+3)}$ & $\frac{B}{(x+3)^2}$

(x^2+3) is a quadratic so it needs a partial fraction $\frac{Cx+D}{x^2+3}$

$$\frac{x^2+15}{(x+3)^2(x^2+3)} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{Cx+D}{(x^2+3)}$$

$$x^2+15 = A(x+3)(x^2+3) + B(x^2+3) + (Cx+D)(x+3)^2$$

method 1: roots

there is a root at $x=-3$

$(-3)^2+15 = A(-3+3)((-3)^2+3) + B((-3)^2+3) + (Cx+D)(-3+3)^2$

$9 + 15 = 0 + 12B + 0$

$24 = 12B$

$2 = B$

the root method will not work as we can't get down to just one of A, B, C method 2: linear algebra

multiple out:

$x^2+15 = A(x^3+3x+3x^2+9) + B(x^2+3) + Cx(x^2+6x+9) + D(x^2+6x+9)$

$x^2+15 = Ax^3 + 3Ax + 3Ax^2 + 9A + Bx^2 + 3B + Cx^3 + 6Cx^2 + 9Cx + Dx^2 + 6Dx + 9D$

$x^2+15 = Ax^3 + Cx^3 + 3Ax^2 + Bx^2 + 6Cx^2 + Dx^2 + 3Ax + 9Cx + 6Dx + 9A + 3B + 9D$

plug in b = 2 from method 1:

$x^2+15 = Ax^3 + Cx^3 + 3Ax^2 + 2x^2 + 6Cx^2 + Dx^2 + 3Ax + 9Cx + 6Dx + 9A + 6 + 9D$

separate the powers:

$x^3: 0x^3 = Ax^3 + Cx^3$

$x^2: 1x^2 = 3Ax^2 + 2x^2 + 6Cx^2 + Dx^2$

$x: 0x = 3Ax + 9Cx + 6Dx$

constants: $15 = 9A + 6 + 9D$

simplify:

$0 = A + C$

$-1 = 3A + 6C + D$

$0 = 3A + 9C + 6D$

$1 = A + D$

subtract eq.4 from eq.2

$0 = A + C$

$-2 = 2A + 6C$

$0 = 3A + 9C + 6D$

$1 = A + D$

Use row 3 to solve

$C = -1/2$

$A = 1/2$

$D = 1/2$

$$\frac{x^2+15}{(x+3)^2(x^2+3)} = \frac{1/2}{x+3} + \frac{2}{(x+3)^2} + \frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+3}$$

Key Partial Fraction Rules

- This method only works for proper fractions, i.e. the denominator is larger.
- To make an improper fraction proper, we use polynomial long division.
- The denominator is made of linear factors and irreducible quadratics.

- The partial fraction of an irreducible quadratic uses $Ax+B$ in the numerator.
- If the factor has an exponent then you need multiple partial fraction, one for each exponent.

Key Denominators and their Partial Fractions

Factor in denominator	Partial fraction in decomposition
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^n$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^n$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$

Examples:

Give the partial fraction decomposition for the following, do not solve for the coefficients:

1. $\frac{x^2+x+1}{(x+1)(x+4)^2} = \frac{A}{x+1} + \frac{B}{(x+4)} + \frac{C}{(x+4)^2}$

2. $\frac{x^2+x+1}{x^3(x-1)(x^4-1)} = \frac{x^2+x+1}{x^3(x-1)(x^2+1)(x^2-1)} = \frac{x^2+x+1}{x^3(x-1)(x^2+1)(x+1)(x-1)} = \frac{x^2+x+1}{x^3(x-1)^2(x^2+1)(x+1)}$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2} + \frac{Fx+G}{x^2+1} + \frac{H}{x+1}$$

3. $\frac{100x^7+x^3+5000}{(x^2+4)^2(x^4+5x^3+6x^3)(16-x^4)} = \frac{100x^7+x^3+5000}{(x^2+4)^2(x^2)(x^2+5x+6)(4-x^2)(4+x^2)} = \frac{100x^7+x^3+5000}{(x^2+4)^2(x)^2(x+3)(x+2)(2+x)(2-x)(4+x^2)}$

$$= \frac{100x^7+x^3+5000}{(x^2+4)^2(x)^2(x+2)^2(2-x)(4+x^2)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2} + \frac{E}{x} + \frac{F}{x^2} + \frac{G}{x+2} + \frac{H}{(x+2)^2} + \frac{I}{2-x} + \frac{Jx+K}{(4+x^2)}$$

Dividing Polynomials

Proper Rational Functions

Definitions:

A rational function is a function of the form $\frac{p(x)}{q(x)}$ where both $p(x)$ and $q(x)$ are polynomials, e.g. $\frac{2}{3}$, $\frac{4x+5}{2x-1}$, $\frac{x+1}{3x^2-2}$, $\frac{x^3-1}{5x-4}$, etc.

A rational function is called proper if the degree of the top is smaller than the degree of the bottom, e.g. $\frac{5}{x^2-4}$, $\frac{5x}{3x^2-1}$, $\frac{x^2+2}{(x-1)(x+3)^2}$

Examples: Identify if the following rational functions are proper:

1. $\frac{1}{x^2-4}$ $\deg(1)=0<2=\deg(x^2-4)$
2. $\frac{x-1}{x^2-4}$ $\deg(x-1)=1<2=\deg(x^2-4)$
3. $\frac{x^2-1}{x^2-4}$ $\deg(x^2-1)=2\not<2=\deg(x^2-4)$
4. $\frac{x^3-1}{x^2-4}$ $\deg(x^3-1)=3\not<2=\deg(x^2-4)$
5. $\frac{x^3-1}{(x^2-4)^2}$ $\deg(x^3-1)=3<4=\deg((x^2-4)^2)$

proper: 1, 2, 5

Long Division

Similar to how we can use long division to turn an improper fraction (like $\frac{24}{5}$) into a proper fraction ($4+\frac{4}{5}$), we can use long division on polynomials

Examples:

1. Divide $5x^3-x^2+6$ by $x-4$

$$
\begin{array}{rl}
 & 5x^2+19x+76 \\
x-4 \,\big) & 5x^3-x^2+6 \\
 & -(5x^3-20x^2) \\
\hline
 & 0+19x^2+6 \\
 & -(19x^2-76x) \\
\hline
 & 0+76x+6 \\
 & -(76x-304) \\
\hline
 & 0+310
\end{array}
$$

we have one x in $(x-4)$ we need $5x^3$ so we subtract $5x^2(x-4)$

we have one x we need $19x^2$ so we subtract $19x(x-4)$

we have one x we need $76x$ so we subtract $76(x-4)$

we have one x we need 310 which is impossible so 310 is our remainder

$$\frac{5x^2+19x+76}{x-4} = 5x^2+19x+76+\frac{310}{x-4}$$