1. The linear mass density of a rod is given by $\rho(x) = \dfrac{2x+1}{\sqrt{4-x^2}}$kg/m for $0 \le x \le 1$. Find the total mass of the rod in kg.

(linear) mass $= \int_a^b \rho(r)\,dr$

$$\text{mass} = \int_0^1 \frac{2x+1}{\sqrt{4-x^2}}\,dx$$

$$= \int_0^1 \frac{2x}{\sqrt{4-x^2}} + \frac{1}{\sqrt{4-x^2}}\,dx$$

$$= \int_0^1 \frac{2x}{\sqrt{4-x^2}} + \frac{1}{2\sqrt{1-(\frac{x}{2})^2}}\,dx$$

$u = 4-x^2$ $\qquad$ $v = \frac{x}{2}$

$du = -2x\,dx$ $\qquad$ $dv = \frac{1}{2}dx$

$$= \int_4^3 \frac{-1}{\sqrt{u}}\,du + \int_0^{1/2} \frac{1}{\sqrt{1-v^2}}\,dv$$

$$= -2u^{1/2}\Big|_4^3 + \arcsin^{-1}(v)\Big|_0^{1/2}$$

$$= -2(3)^{1/2} - (-2(4)^{1/2}) + \arcsin^{-1}(1/2) - \arcsin(0)$$

$$= -2\sqrt{3} + 2\cdot(2) + \frac{\pi}{6} - 0$$

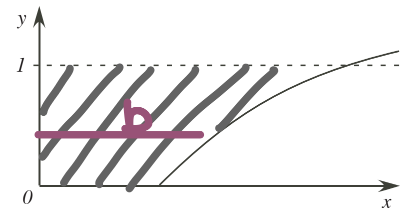
$$= 4 + \frac{\pi}{6} - 2\sqrt{3}$$

2a. Consider the solid whose base is bounded by the lines $y = \ln x$, $y = 1$, x-axis and y-axis. Find the volume of the solid if the cross-sections perpendicular to the y-axis are (a) squares, (b) semi-circles, and (c) triangles of height y?

$$V = \int_a^b A(y)\,dy$$

$y = \ln(x)$

$e^y = x$



(a) squares

$$A(y) = b^2$$
$$= (e^y)^2$$

$$V = \int_0^1 (e^y)^2\,dy$$
$$= \int_0^1 e^{2y}\,dy$$
$$= \frac{1}{2}e^{2y}\Big|_0^1$$
$$= \frac{1}{2}e^2 - \frac{1}{2}$$

(b) semicircles

$$A(y) = \frac{1}{2}\pi r^2$$
$$= \frac{1}{2}\pi\left(\frac{1}{2}e^y\right)^2$$

$$V = \int_0^1 \frac{1}{8}\pi e^{2y}\,dy$$
$$= \frac{1}{16}\pi e^{2y}\Big|_0^1$$
$$= \frac{1}{16}\pi\left[e^2 - 1\right]$$

(c) triangles

$$A(y) = \frac{1}{2}bh$$
$$= \frac{1}{2}(e^y)(y)$$

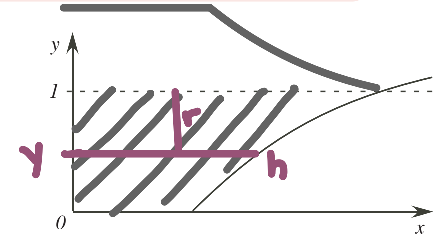
$$V = \int_0^1 \frac{1}{2}ye^y\,dy$$

$u = y \qquad dv = e^y dy$

$du = 1dy \qquad v = e^y$

$$= \frac{1}{2}\left[ye^y - \int_0^1 e^y\,dy\right]$$
$$= \frac{1}{2}\left[ye^y - e^y\Big|_0^1\right]$$
$$= \frac{1}{2}\left[(e^1 - e^1) - (0 - 1)\right]$$
$$= \frac{1}{2}[1]$$
$$= \frac{1}{2}$$

2b. (Not Related to the Above) Consider the region bounded by the lines $y = \ln x$, $y = 1$, x-axis and y-axis. Find the volume of the solid formed when the region is revolved about the line $y = 1$.



disk: $V = \int_a^b \pi r^2 \, dx$

fails bc two radii

shell: $V = \int_a^b 2\pi r h \, dy$

$$= \int_0^1 2\pi (1-y)(e^y)\,dy$$

$$= 2\pi \int_0^1 e^y - ye^y \, dy$$

$u = y \quad dv = e^y dy$

$du = dy \quad v = e^y$

$$= 2\pi \left[e^y - \left(ye^y - \int_0^1 e^y dy \right) \right]$$

$$= 2\pi \left[e^y - ye^y + e^y \Big|_0^1 \right]$$

$$= 2\pi \left[(e^1 - e^1 + e^1) - (1 - 0 + 1) \right]$$

$$= 2\pi [e - 2]$$

3. A **20** meter long tank has cross section as shown in Figure 1 below. All dimensions are in meters. If the tank is **completely** filled with fluid of density 3000 kg/m^3, what is the work (in Joules) required to pump all the fluid to the top of the tank. You may assume that the acceleration due to gravity is $g = 10$ m/s^2.

You must show your set-up and method of solution to obtain credit.

$\Delta W = \text{density} \cdot \text{volume} \cdot \text{gravity} \cdot \text{displ.}$

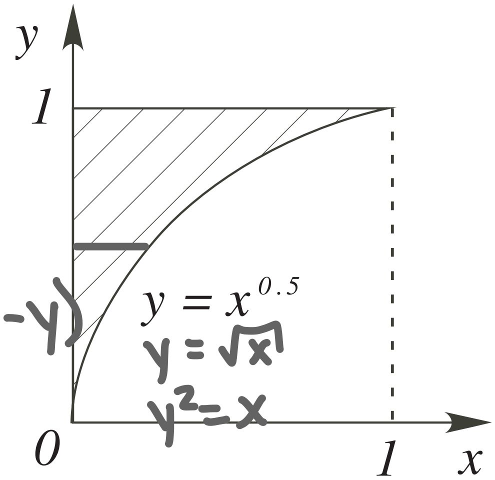


Figure 1.

$$\Delta W = 3000 \cdot \underbrace{y^2 \cdot 20 \cdot \Delta y}_{b \cdot w \cdot h} \cdot 10 \cdot (1-y)$$

$$W = \int_0^1 600000\, y^2(1-y)\,dy$$

$$= 600000 \int_0^1 y^2 - y^3\,dy$$

$$= 600\,000 \left[\tfrac{1}{3}y^3 - \tfrac{1}{4}y^4\right]_0^1$$

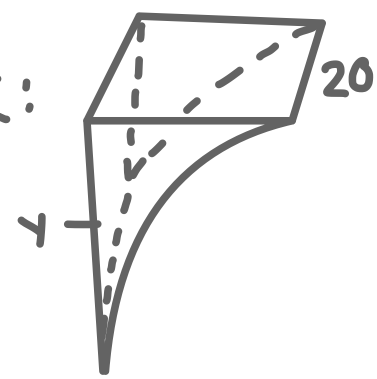
$$= 600\,000 \left[\tfrac{1}{3} - \tfrac{1}{4}\right]$$

$$= 600\,000 \left[\tfrac{1}{12}\right]$$

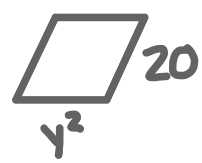
$$= \frac{200\,000}{4}$$

$$= 50000$$

tank:



slice:



4a. What is the rate equation for an amount $y(t)$ of substance growing or decaying exponentially (with respect to time t)? What is the formula for $y(t)$? Explain any constants used in your answer.

$$\frac{dy}{dt} = ky \qquad k = \text{growth constant}$$

$$A(t) = A_0 e^{kt} \qquad A_0 = \text{initial value}$$

4b. If 80% of a radioactive substance remains after 4 hours, how long does it take for the amount to reduce to 30% of its initial quantity? You may use the fact that the radioactive substance decays exponentially.

$$A(t) = A_0 e^{rt}$$

$$0.8\cancel{A_0} = \cancel{A_0} e^{r \cdot 4}$$

$$\ln(0.8) = 4r$$

$$\frac{1}{4}\ln(0.8) = r$$

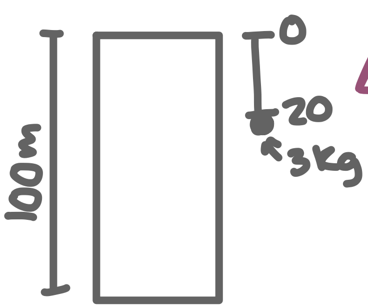
$$A(t) = A_0 e^{\frac{1}{4}\ln(0.8)t}$$

$$0.3\cancel{A_0} = \cancel{A_0} e^{\frac{1}{4}\ln(0.8)t}$$

$$\ln(0.3) = \frac{1}{4}\ln(0.8)t$$

$$4\frac{\ln(0.3)}{\ln(0.8)} = t$$

5. (Not Related to the Above) A 20 meter **uniform** chain with mass 100 kg hangs on the side of a 100 meter tall building. If a 3 kg weight is hung at the lower end of the chain, find the work done to pull the whole chain and the weight to the top of the building.



$$\Delta W = \text{density} \cdot \text{volume} \cdot \text{gravity} \cdot \text{displ.}$$

$$\Delta W_{chain} = \left(\frac{100}{20}\right)(\Delta y)(10)(y)$$

$$W_{chain} = \int_0^{20} 50y\,dy$$

$$= 25y^2 \Big|_0^{20}$$

$$= 10000$$

$$W_{ball} = (3)(10)(20) = 600$$

$$W = 10000 + 600 = 10600$$

6a. Set up but not evaluate the integral that gives the volume of the solid obtained when the area between $y = x^2 + 1$ and the x-axis over $0 \le x \le 1$ is revolved about the line $y = 2$.

$$V = \int_a^b \pi (R^2 - r^2)\,dx$$

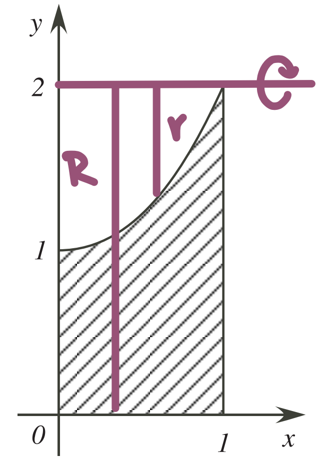
$$= \int_0^1 \pi \left((2-0)^2 - (2-(x^2+1))^2\right)dx$$

$$= \int_0^1 \pi \left(4 - (2-x^2-1)^2\right)dx$$

$$= \int_0^1 \pi \left(4 - (-x^2+1)^2\right)dx$$

$$= \int_0^1 \pi \left(4 - (1-2x^2+x^4)\right)dx$$

$$= \int_0^1 \pi (3+2x^2-x^4)\,dx$$



6b. Set up but not evaluate the integral that gives the volume of the solid form when the same region in (a) is revolved about the $x = 2$?

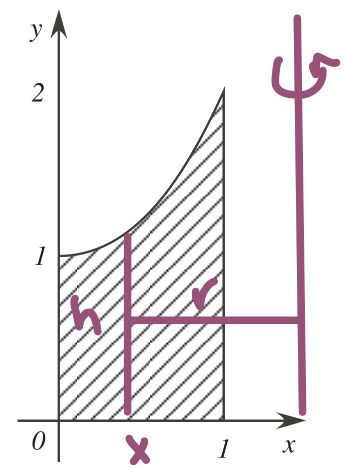
washer fails bc 2 radii

shell: $V = \int_a^b 2\pi r h \, dx$

$$= \int_0^1 2\pi (2-x)(x^2+1)\,dx$$

$$= 2\pi \int_0^1 2x^2 + 2 - x^3 - x \; dx$$

$$= 2\pi \int_0^1 2 - x + 2x^2 - x^3 \, dx$$



7. The density of a **circular** disc of radius 1 meter is given by the radial function

$$\rho(r) = \frac{1}{r(4+r^2)},$$

where r is the distance from the center of the disc, and ρ is in kilograms per m^2. Find the total mass of the disc in kilograms.

(radial) mass $= \int_a^b 2\pi r\, \rho(r)\, dr$

$$\text{mass} = \int_0^1 2\pi r \frac{1}{r(4+r^2)}\, dr$$

$$= \int_0^1 2\pi \frac{1}{4+r^2}\, dr$$

$$= \int_0^1 2\pi \frac{1}{4(1+(\frac{1}{2}r)^2)}\, dr$$

$$= \int_0^1 \frac{1}{2}\pi \frac{1}{1+(\frac{1}{2}r)^2}\, dr$$

$$= \frac{1}{2}\pi \arctan(\tfrac{1}{2}r)\cdot 2 \Big|_0^1$$

$$= \pi \arctan(\tfrac{1}{2}r) \Big|_0^1$$

$$= \pi[\arctan(\tfrac{1}{2}) - \arctan(0)]$$

$$= \pi[\arctan(\tfrac{1}{2}) - 0]$$

$$= \pi \cdot \arctan(\tfrac{1}{2})$$

10360 Extra Review Exam01 Name ______________________

1. Find the derivative of the following functions:

1a. $y = (e^2 + 5x)^3$

power rule + chain

$y' = 3(e^2+5x)^2 \cdot 5$

1b. $y = e^{x^2+(\ln x)^2}$

exponential + chain + power x2 + chain + ln

$y' = e^{x^2+(\ln(x))^2} \cdot \left(2x + 2(\ln(x))' \cdot \frac{1}{x}\right)$

1c. $y = x^{arcsin(x)}$

logarithmic differentation

$y = arsin(x) \cdot \ln(x)$

$\frac{1}{y} y' = \frac{1}{\sqrt{1-x^2}} \cdot \ln(x) + \frac{1}{x} \arcsin(x)$

$y' = \left(x^{\arcsin(x)}\right)\left[\frac{1}{\sqrt{1-x^2}} \cdot \ln(x) + \frac{1}{x} \arcsin(x)\right]$

2. The slope of the graph of $y = f(x)$ is given by

$$\frac{5-9x}{\sqrt{16-9x^2}}.$$

Find a formula for $f(x)$ if the graph passes through the point $(0, 2)$.

initial value problem

$$\int \frac{5-9x}{\sqrt{16-9x^2}}\,dx$$

$$= \int \frac{5}{\sqrt{16-9x^2}} - \frac{9x}{\sqrt{16-9x^2}}\,dx$$

$$= \int \frac{5}{4}\cdot\frac{1}{\sqrt{1-(\frac{3}{4}x)^2}} - 9x(16-9x^2)^{-1/2}\,dx$$

power rule ↓ u-sub

$$= \frac{5}{4}\arcsin\left(\tfrac{3}{4}x\right)\cdot\frac{4}{3} - 9(16-9x^2)^{1/2}\cdot 2\cdot\frac{1}{-18} + C$$

$$f(x) = \frac{5}{3}\arcsin\left(\tfrac{3}{4}x\right) + (16-9x^2)^{1/2} + C$$

$$f(0) = \frac{5}{3}\arcsin(0) + (16)^{1/2} + C$$

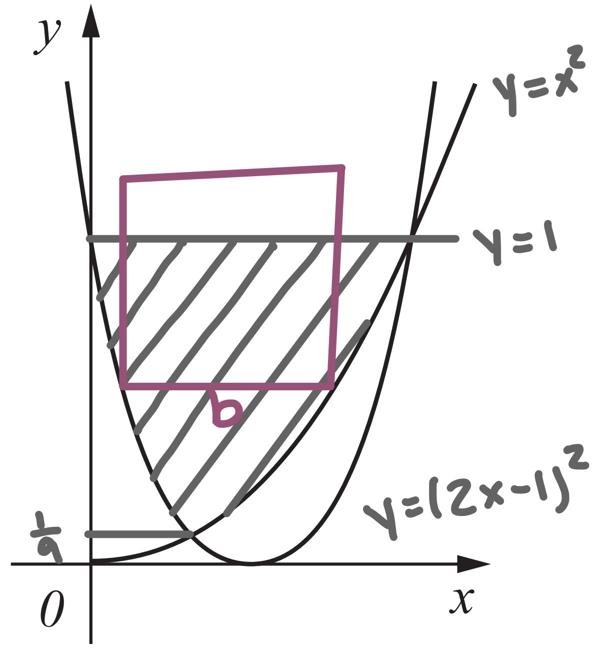
$$2 = \frac{5}{3}(0) + 4 + C$$

$$-2 = C$$

$$f(x) = \frac{5}{3}\arcsin\left(\tfrac{3}{4}x\right) - (16-9x^2)^{1/2} - 2$$

3. Consider the solid whose base is the finite region between $y = x^2$, $y = (2x-1)^2$, and $y = 1$. If the slices perpendicular to y-axis are square, find the volume of the solid.

$V = \int_a^b A(y)\,dy$



$A(y) = b^2$

$= \left(\sqrt{y} - \left(\frac{-\sqrt{y}+1}{2}\right)\right)^2$

$V = \int_{1/9}^{1} \left(\sqrt{y} + \frac{1}{2}\sqrt{y} - \frac{1}{2}\right)^2 dy$

$= \int_{1/9}^{1} \left(\frac{3}{2} y^{1/2} - \frac{1}{2}\right)^2 dy$

$= \int_{1/9}^{1} \left(\frac{3y^{1/2} - 1}{2}\right)^2 dy$

$= \int_{1/9}^{1} \frac{1}{4}(3y^{1/2} - 1)^2 dy$

$= \frac{1}{4}\int_{1/9}^{1} 9y - 6y^{1/2} + 1\,dy$

$= \frac{1}{4}\left[\frac{9}{2}y^2 - 6\cdot\frac{2}{3}y^{3/2}\right]_{1/9}^{1}$

$= \frac{1}{4}\left[\frac{9}{2} - 4 - \left(\frac{1}{2}\cdot\frac{1}{9} - 4(1/9)^{3/2}\right)\right]$

$= \frac{1}{4}\left[\frac{9}{2} - 4 - \frac{1}{8} - 4(1/9)^{3/2}\right]$

$y = x^2 \Rightarrow \sqrt{y} = x$

$y = (2x-1)^2$

$\Rightarrow \frac{-\sqrt{y}+1}{2} = x$

$x^2 = (2x-1)^2$

$x^2 = 4x^2 - 4x + 1$

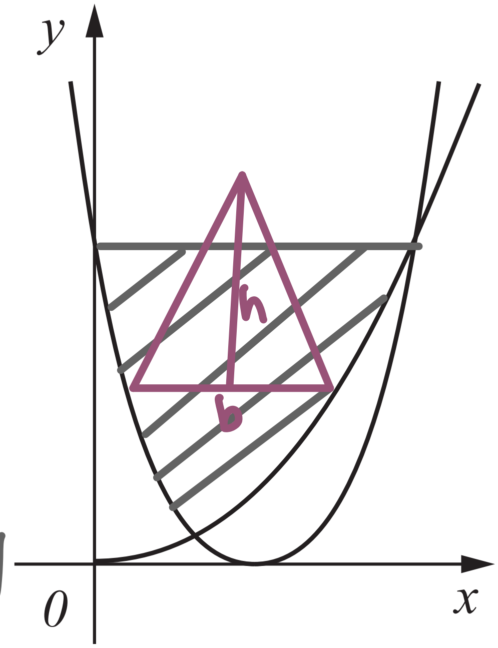
$0 = 3x^2 - 4x + 1$

$0 = (x-1)(3x-1)$

$x = 1, \frac{1}{3}$

$y = x^2 = \left(\frac{1}{3}\right)^2$

4. Consider the solid whose base is the finite region between $y = x^2$, $y = (2x-1)^2$, and $y = 1$. If the slices perpendicular to y-axis are triangles of height y^2, find the volume of the solid.



$$V = \int_a^b A(y)\,dy$$

$$= \int_{1/9}^{1} \frac{1}{2} b h \, dy$$

$$= \int_{1/9}^{1} \frac{1}{2}\left(\frac{3y^{1/2}-1}{2}\right)(y^2)\,dy$$

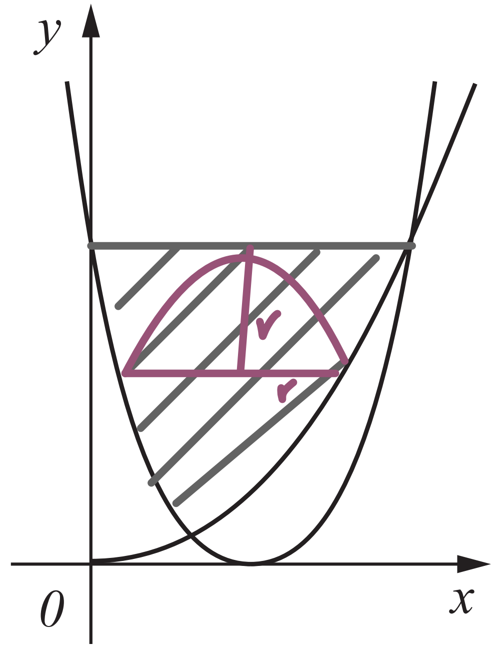
$$= \frac{1}{4}\int_{1/9}^{1} 3y^{3/2} - y^2 \, dy$$

$$= \frac{1}{4}\left[3\cdot\frac{2}{5}y^{5/2} - \frac{1}{3}y^2\right]_{1/9}^{1}$$

$$= \frac{1}{4}\left[\frac{6}{5} - \frac{1}{3} - \left(\frac{6}{5}\left(\frac{1}{9}\right)^{5/2} - \frac{1}{3}\left(\frac{1}{9}\right)^2\right)\right]$$

$$= \frac{1}{4}\left[\frac{6}{5} - \frac{1}{3} - \frac{6}{5}(1/9)^{5/2} - \frac{1}{243}\right]$$

5. Consider the solid whose base is the finite region between $y = x^2$, $y = (2x-1)^2$, and the y-axis. If the slices perpendicular to x-axis are semi-circles, find the volume of the solid.



$$V(x) = \int_a^b A(y)\,dy$$

$$= \int_{1/9}^{1} \frac{1}{2}\pi r^2\,dy$$

$$= \int_{1/9}^{1} \frac{1}{2}\pi\left(\frac{1}{2}\left(\frac{3y^{1/2}-1}{2}\right)\right)^2 dy$$

$$= \int_{1/9}^{1} \frac{1}{2}\pi\left(\frac{1}{4}(3y^{1/2}-1)\right)^2 dy$$

$$= \int_{1/9}^{1} \frac{1}{2}\pi\left(\frac{1}{16}\right)(3y^{1/2}-1)^2 dy$$

$$= \frac{1}{32}\pi\int_{1/9}^{1} (3y^{1/2}-1)^2 dy$$

$$= \frac{1}{32}\pi\int_{1/9}^{1} 9y - 6y^{1/2} + 1\,dy$$

$$= \frac{1}{32}\pi\left[\frac{9}{2}y^2 - 6\cdot\frac{2}{3}y^{3/2} + y\right]_{1/9}^{1}$$

$$= \frac{1}{32}\pi\left[\frac{9}{2} - 4 + 1 - \frac{1}{18} + 4(1/9)^{3/2} - \frac{1}{9}\right]$$

6. A 50 meter long 100kg uniform chain is dangled from the top of a 200 m building. What is the work done to spool the chain to the top of the building? What is the work done is only 30 m is spooled to the top with the rest left dangling?

$\Delta W =$ density $\cdot$ volume $\cdot$ gravity $\cdot$ disp.

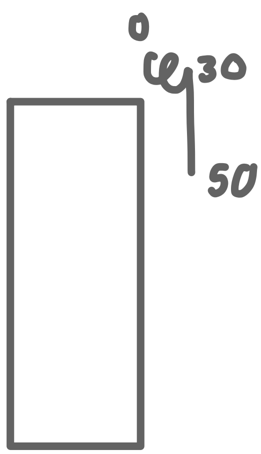
(a) $\Delta W = \left(\frac{100}{50}\right) \cdot \Delta y \cdot 10 \cdot y$

$$W = \int_0^{50} 20y\,dy$$

$$= 10y^2 \Big|_0^{50}$$

$$= 10(2500)$$

$$= 25000$$



(b) $W_{upper} = \int_0^{30} 20y\,dy$

$$= 10y^2 \Big|_0^{30}$$

$$= 10(900)$$

$$= 9000$$

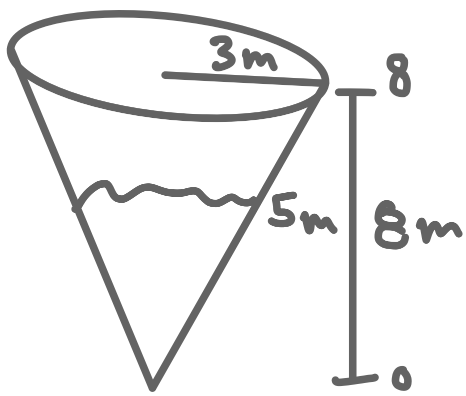
displacement

$$W_{lower} = \int_{30}^{50} 20 \cdot 30\,dy$$

$$= 600y \Big|_{30}^{50}$$

$$= 600(20) = 12000$$

7. A conical tank of radius 3 meter and height 8 meter is filled with water to the depth of 5 meters. Find the work done to pump all the water out from (a) the top of the tank and (b) a sprout with opening 4 meter above the top of the tank.



(a) ΔW = density · volume · gravity · displace.

$$\Delta W = (1000)(\pi r^2)\cdot 10 \cdot (8-y)$$

$$\frac{r}{h} = \frac{3}{8} \Rightarrow r = \frac{3}{8}h$$

$$W = \int_0^5 10000\pi \left(\frac{3}{8}y\right)^2 (8-y)\,dy$$

$$= \frac{9}{64}\cdot 10000\pi \int_0^5 8y^2 - y^3\,dy$$

$$= \frac{9}{64}\cdot 10000\pi \left[\frac{8}{3}y^3 - \frac{1}{4}y^4\right]_0^5$$

(b) add spout = displacement + 4

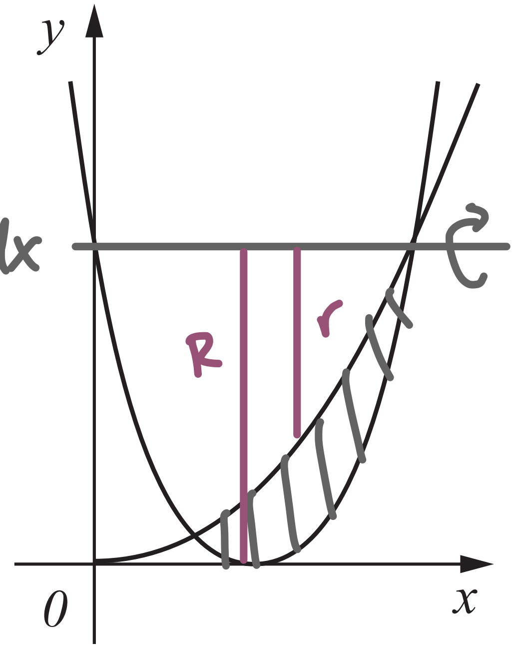
$$W = \int_0^5 \frac{9}{64}\cdot 10000\pi \cdot y^2 (12-y)\,dy$$

$$= \frac{9}{64}\cdot 10000\pi \int_0^5 12y^2 - y^3\,dy$$

8A. Consider the finite region bounded between $y = x^2$, $y = (2x-1)^2$. Find the volume of the solid generated when this region is revolved about $y = 1$.

washer: $V = \int_a^b \pi(R^2 - r^2)\,dr$

$V = \int_{1/3}^{1} \pi \left((1-(2x-1)^2)^2 - (1-x^2)^2 \right) dx$



$x^2 = (2x-1)^2$

$x^2 = 4x^2 - 4x + 1$

$0 = 3x^2 - 4x + 1$

$0 = (x-1)(3x-1)$

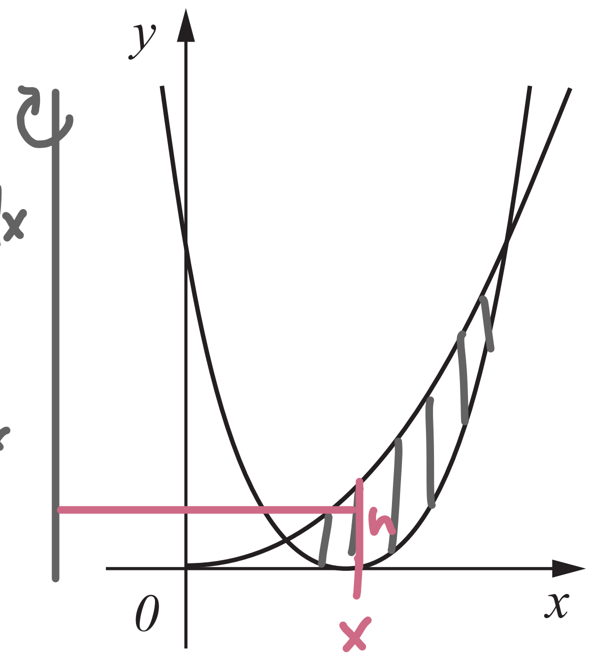
$x = 1, \frac{1}{3}$

8B. Consider the finite region bounded between $y = x^2$, $y = (2x-1)^2$. Find the volume of the solid generated when this region is revolved about $x = -1$.

shell: $V = \int_a^b 2\pi r h \, dx$

$$V = \int_{1/3}^{1} 2\pi (x - (-1))(x^2 - (2x-1)^2) dx$$

$$= \int_{1/3}^{1} 2\pi (x+1)(x^2 - (2x-1)^2) dx$$

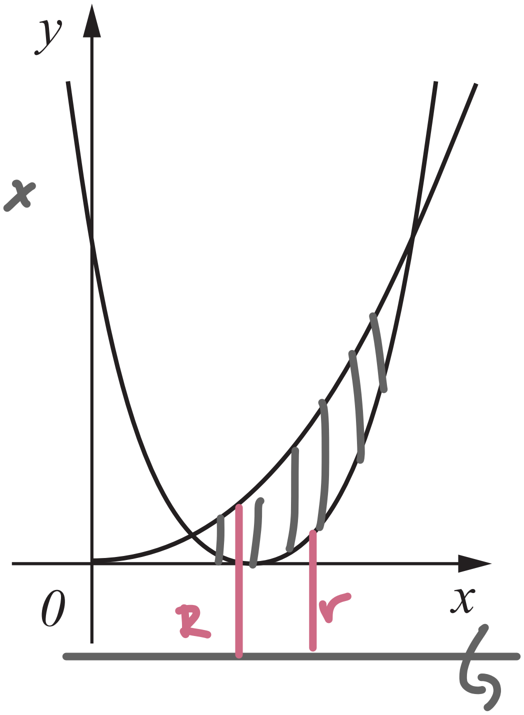


8C. Consider the finite region bounded between $y = x^2$, $y = (2x-1)^2$. Find the volume of the solid generated when this region is revolved about $y = -1$.

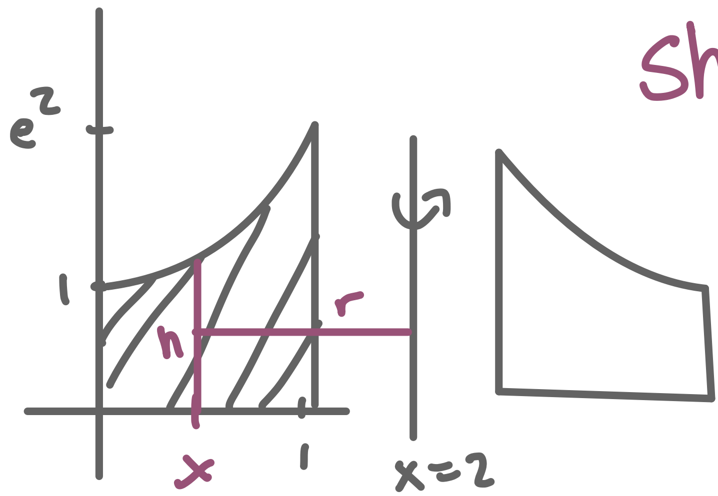
washer: $V = \int_a^b \pi (R^2 - r^2)\, dx$

$$V = \int_{1/3}^{1} \pi \left((x^2 - (-1))^2 - ((2x-1)^2 - (-1))^2 \, dx \right.$$

$$= \int_{1/3}^{1} \pi \left((x^2+1)^2 - ((2x-1)^2+1)^2 \, dx \right.$$



9. Find the volume of the solid when the area under the graph $y = e^{2x}$ for $0 \le x \le 1$ is revolved about the line $x = 2$.



Shell: $V = \int_a^b 2\pi r h \, dx$

$$V = \int_0^1 2\pi (2-x)(e^{2x})\,dx$$

$$= 2\pi \int_0^1 (2-x) e^{2x}\,dx$$

$u = 2-x \qquad dv = e^{2x}$

$du = -dx \qquad v = \frac{1}{2}e^{2x}$

$$= 2\pi\left[(2-x)\cdot\frac{1}{2}e^{2x}\Big|_0^1 + \int \frac{1}{2}e^{2x}\,dx\right]$$

$$= 2\pi\left(\frac{1}{2}(2-x)e^{2x} + \frac{1}{4}e^{2x}\right)\Big|_0^1$$

$$= 2\pi\left(\frac{1}{2}e^2 + \frac{1}{4}e^2 - 1 - \frac{1}{4}\right)$$

$$= 2\pi\left(\frac{3}{4}e^2 - \frac{5}{4}\right)$$