Improper Integrals

Infinite Integral

In an infinite integral one, or both, of the limits of integration are infinite. We have solved integrals over definite intervals, but these are integrals over infinite intervals. Consider the integral $\int_1^\infty \frac{1}{x^2}\,dx$.

This is an integral we have seen before, but because of the infinity we can not just integrate and "plug in." Let us recall what an integral is: the area under the curve $f(x) = \frac{1}{x^2}$ over the interval $[1, \infty]$.

This is still hard to compute, so instead we consider the area under the curve $f(x) = \frac{1}{x^2}$ over the interval $[1, t]$ where $t > 1$ and t is finite. This is something we can do: $A_t = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^t = 1 - \frac{1}{t}$.

Now we can consider the area under $f(x)$ on $[1, \infty)$ simply by taking the limit of A_t as t goes to infinity: $A_\infty = \lim_{t\to\infty} A_t = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$.

This is how we tackle the integral itself,

$$\begin{aligned}
\int_1^\infty \frac{1}{x^2}\,dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx \\
&= \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t \\
&= \lim_{t\to\infty} \left[1 - \frac{1}{t}\right] \\
&= 1.
\end{aligned}$$

$\lim_{t\to\infty}\left(\frac{1}{t}\right) \approx \frac{1}{\text{infinitely big}} \approx \text{infinitely small} \approx 0$

This is how we deal with these kinds of integrals in general. We will replace the infinity with a constant (I like c), do the integral and take the limit of the result as the constant goes to infinity.

In this example the area under the curve over an infinite interval was not infinity as one might expect. Instead we got a small number. This isn't always the case. We call these integrals convergent if the limit exists and is a finite number and divergent if the limit either doesn't exist or is (plus or minus) infinity.

1. If $\int_a^t f(x)\,dx$ exists for every $t > a$ then $\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$ provided the limit exists and is finite.
2. If $\int_t^b f(x)\,dx$ exists for every $t < b$ then $\int_{-\infty}^b f(x)\,dx = \lim_{t\to\infty}\int_t^b f(x)\,dx$ provided the limit exists and is finite.

3. If $\int_{-\infty}^{c} f(x)\,dx$ and $\int_{c}^{\infty} f(x)\,dx$ are both convergent then,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$ where c is any number.

Note that both integrals must converge for this integral to be convergent.

Examples:

1. Determine if the following integral is convergent or divergent. If convergent, find its value.

(a) $\int_{1}^{\infty} \frac{1}{x}\,dx$

$= \lim_{t\to\infty} \int_{1}^{t} \frac{1}{x}\,dx$

$= \lim_{t\to\infty} [\ln|x|]_{1}^{t}$

$= \lim_{t\to\infty} [\ln(t) - \ln(1)]$

$= \infty$

$\therefore$ divergent

(b) $\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}\,dx$

$= \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{\sqrt{3-x}}\,dx$

$= \lim_{t\to-\infty} [-2\sqrt{3-x}]_{t}^{0}$

$= \lim_{t\to-\infty} (-2\sqrt{3} + 2\sqrt{3-t})$

$= -2\sqrt{3} + \infty = \infty$

$\therefore$ divergent

(c) $\int_{-\infty}^{\infty} x e^{-x^2}\,dx$

$= \int_{-\infty}^{0} x e^{-x^2}\,dx + \int_{0}^{\infty} x e^{-x^2}\,dx$

$= \lim_{t\to-\infty} \int_{t}^{0} x e^{-x^2}\,dx + \lim_{s\to\infty} \int_{0}^{s} x e^{-x^2}\,dx$

$= \lim_{t\to-\infty} [-\frac{1}{2} e^{-x^2}]_{t}^{0} + \lim_{s\to\infty} [-\frac{1}{2} e^{-x^2}]_{0}^{s}$

$= \lim_{t\to-\infty} [-\frac{1}{2} + \frac{1}{2} e^{-t^2}] + \lim_{s\to\infty} [-\frac{1}{2} e^{-s^2} + \frac{1}{2}]$

$= -\frac{1}{2} + \frac{1}{2}$

$= 0$

$\therefore$ convergent

(c) $\int_{-2}^{\infty} \sin(x)\,dx$

$= \lim_{t\to\infty} \int_{-2}^{t} \sin(x)\,dx$

$= \lim_{t\to\infty} [-\cos(x)]_{-2}^{t}$

$= \lim_{t\to\infty} [\cos(2) - \cos(t)]$

limit does not exists!

$\therefore$ divergent

Discontinuous Integrand

The second type of improper integral is a discontinuous integral.

1. If $f(x)$ is continuous on the interval $[a,b)$ and not continuous at $x=b$ then $\int_{a}^{b} f(x)\,dx = \lim_{t\to b^-} \int_{a}^{t} f(x)\,dx$ provided the limit exists and is finite.

2. If $f(x)$ is continuous on the interval $(a,b]$ and not continuous at $x=a$ then $\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$ provided the limit exists and is finite.

Note as well that we use one-sided limits here since the interval of integration is entirely on the one side. Upper = left, lower = right.

3. If $f(x)$ is not continuous at $x=c$ where $a<c<b$ and $\int_a^c f(x)dx$ & $\int_c^b f(x)dx$ are both convergent then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.
Note that both integrals must be convergent for the total to be convergent.

4. If $f(x)$ is not continuous at $x=a$ and $x=b$ and if $\int_a^c f(x)dx$ & $\int_c^b f(x)dx$ are both convergent then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where c is any number $a<c<b$. Again, both must be convergent for the total to be.

Examples:

2. Determine if the following integral is convergent or divergent. If convergent, find its value.

(a) $\int_0^3 \frac{1}{\sqrt{3-x}}dx$

$= \lim_{t\to 3^-} \int_0^t \frac{1}{\sqrt{3-x}}dx$

$= \lim_{t\to 3^-} \left[-2\sqrt{3-x}\right]_0^t$

$= \lim_{t\to 3^-} \left(2\sqrt{3} - 2\sqrt{3-t}\right)$

$= 2\sqrt{3}$ $\therefore$ converge

(b) $\int_{-2}^3 \frac{1}{x^3}dx$

$= \int_{-2}^0 \frac{1}{x^3}dx + \int_0^3 \frac{1}{x^3}dx$

$= \lim_{t\to 0^-} \int_{-2}^t \frac{1}{x^3}dx + \lim_{s\to 3^+} \int_s^3 \frac{1}{x^3}dx$

$= \lim_{t\to 0^-} \left[-\frac{1}{2x^2}\right]_{-2}^t + \lim_{s\to 3^+} \left[-\frac{1}{2x^2}\right]_s^3$

$= \lim_{t\to 0^-} \left(\frac{1}{8} - \frac{1}{2t^2}\right) + \lim_{s\to 3^+} \left(\frac{1}{18} - \frac{1}{2s^2}\right)$

$= -\infty + -\infty$

$= -\infty$ $\therefore$ diverge

(c) $\int_0^\infty \frac{1}{x^2}dx$

$= \int_0^1 \frac{1}{x^2}dx + \int_1^\infty \frac{1}{x^2}dx$

$= \lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}dx + \lim_{s\to\infty} \int_1^s \frac{1}{x^2}dx$

$= \lim_{t\to 0^+} \left[-\frac{1}{x}\right]_t^1 + \lim_{s\to\infty} \left[-\frac{1}{x}\right]_1^s$

$= \lim_{t\to 0^+} \left[-1 + \frac{1}{t}\right] + \lim_{s\to\infty} \left[-1 + \frac{1}{s}\right]$

$= \infty + 0 = \infty$ $\therefore$ diverge