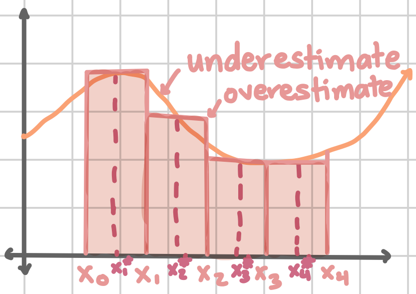
Numerical Integration

So far we have seen integrals that we can compute, but sometimes we run into integrals that we can not compute. The most popular example of this is $\int_0^2 e^{x^2}dx$. Instead of computing the integral, we aim to estimate the values of such definite integrals. Commonly this is done by estimating the area under the curve using shapes we know the area of.

Midpoint Rule

This rule should be familar to you from calculus I (or calculus A). We divide the interval $[a,b]$ into n subintervals of equal width, $\Delta x = \frac{b-a}{n}$, we denote these subintervals by $[x_0,x_1],[x_1,x_2],\ldots,[x_{n-1},x_n]$ where $x_0=a$ and $x_n=b$.

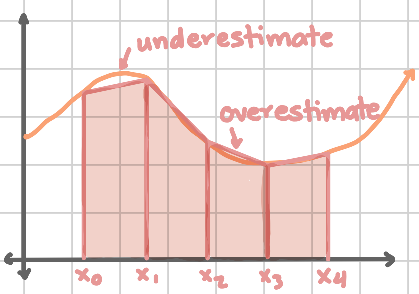


For each interval let x_i^* be the midpoint of the interval. We then sketch rectangles for each subinterval with a height of $f(x_i^*)$. The image shows an example with $n=4$. We can easily find the area for each of these rectangles, the general formula is

$$\int_a^b f(x)dx \approx \Delta x\cdot f(x_1^*) + \Delta x\cdot f(x_2^*) + \ldots + \Delta x\cdot f(x_n^*) = \Delta x\left[f(x_1^*)+f(x_2^*)+\ldots+f(x_n^*)\right].$$

Trapezoid Rule

We once again split our interval $[a,b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$.



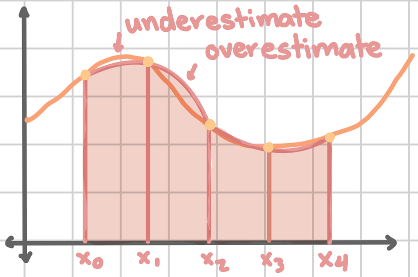
For each subinterval we draw a straight line that is equal to the function values at either endpoint of the interval. The image shows an example of $n=4$. You will notice that the resulting shapes are trapezoids, hence the name of the rule. We can now use the area formula $A_i = \frac{\Delta x}{2}(f(x_{i-1})+f(x_i))$. We add up each trapezoid to find the general formula

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}(f(x_0)+f(x_1)) + \frac{\Delta x}{2}(f(x_1)+f(x_2)) + \ldots + \frac{\Delta x}{2}(f(x_{n-1})+f(x_n))$$

$$= \frac{\Delta x}{2}\left[f(x_0)+2\cdot f(x_1)+2\cdot f(x_2)+\ldots+2\cdot f(x_{n-1})+f(x_n)\right].$$

Simpson's Rule

The final method requires the number of subintervals, n, to be even. The width of each subinterval will still be $\Delta x = \frac{b-a}{n}$. The necessity of n being even will be obvious in a bit.



Unlike the trapezoid rule, which used a straight line approximation, Simpson's rule approximates the function using a quadratic that agrees with 3 points of the function from our subintervals. The image shows an example with $n=4$. The three points are colored differently to help see why we need an even number of subintervals. We can now utilize the area of these approximations, $A_i = \frac{\Delta x}{3}(f(x_{i-1}) + 4f(x_i) + f(x_{i+1}))$ to approximate the integral,

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}(f(x_0)+4f(x_1)+f(x_2)) + \ldots + \frac{\Delta x}{3}(f(x_{n-2})+4f(x_{n-1})+f(x_n))$$
$$= \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+\ldots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)]$$

Notice that all of the odd subscripts are multiplied by 4 and all of the even (except first and last) are multiplied by 2.

Examples:

1. Use $n=4$ and all three rules to approximate the value of the following integral: $\int_0^2 e^{x^2}\,dx$.

   In each case the width of the subintervals will be $\Delta x = \frac{2-0}{4} = \frac{1}{2}$, i.e. $[0,0.5]$, $[0.5,1]$, $[1,1.5]$, $[1.5,2]$.

   Midpoint: $\int_0^2 e^{x^2}dx \approx \frac{1}{2}(e^{(0.25)^2} + e^{(0.75)^2} + e^{(1.25)^2} + e^{(1.75)^2}) = 14.4856$

   Trapezoid: $\int_0^2 e^{x^2}dx \approx \frac{1/2}{2}(e^{(0)^2} + 2e^{(0.5)^2} + 2e^{(1)^2} + 2e^{(1.5)^2} + e^{(2)^2}) = 20.6446$

   Simpson's: $\int_0^2 e^{x^2}dx \approx \frac{1/2}{3}(e^{(0)^2} + 4e^{(0.5)} + 2e^{(1)^2} + 4e^{(1.5)^2} + e^{(2)^2}) = 17.3536$

2. Find $f(5)$ using trapezoid rule with $n=4$ and $f(1)=1$.

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
f'(x)	3.0	3.5	2.0	1.5	2.5	2.0	3.0	3.0	2.5

Recall the fundamental theorem of calculus, $\int_a^b f'(x)\,dx = f(b) - f(a)$ so $f(b) = f(a) + \int_a^b f'(x)\,dx$.

$$f(5) = f(1) + \int_0^5 f'(x)dx$$
$$\approx f(1) + \frac{\Delta x}{2}[f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)]$$
$$= f(1) + \frac{1}{2}[3.0 + 2(2.0) + 2(2.5) + 2(3.0) + 2.5]$$
$$= 1 + \frac{1}{2}[3 + 4 + 5 + 6 + 2.5]$$
$$= 1 + \frac{1}{2}[20.5]$$
$$= 1 + 10.25 = 11.25$$

Exit Ticket Trigonometric Integrals

Trigonometric Integrals Given

$$\int_a^b \sin^n(x)\cos^m(x)dx.$$

if n is odd keep 1 $\sin(x)$ and replace the rest with $\cos(x)$ using $\sin^2(x)+\cos^2(x)=1$ then use $u=\cos(x)$
if m is odd keep 1 $\cos(x)$ and replace the rest with $\sin(x)$ using $\sin^2(x)+\cos^2(x)=1$ then use $u=\sin(x)$
if m and n are odd choose the one with the smallest exponent and follow that path **if m and n are even** utilize half-angle and double-angle formulas

Solve the following integrals using the concept above:

1. $\int \sin^5(x)\cos^7(x)dx$

$= \int \sin(x)(\sin^2(x))^2\cos^7(x)dx$

$= \int \sin(x)(1-\cos^2(x))^2\cos^7(x)dx$

$u=\cos(x)$ $\quad du=-\sin(x)dx$

$= -\int (1-u^2)^2 u^7 du$

$= -\int u^7 - 2u^9 + u^{11} du$

$= -\frac{1}{8}u^8 + \frac{1}{5}u^{10} - \frac{1}{12}u^{12} + C$

$= -\frac{1}{8}\cos^8(x) + \frac{1}{5}\cos^{10}(x) - \frac{1}{12}\cos^{12}(x) + C$

2. $\int \cos^3(x)dx$

$= \int \cos(x)\cos^2(x)dx$

$= \int \cos(x)(1-\sin^2(x))dx$

$u=\sin(x)$ $\quad du=\cos(x)dx$

$= \int 1-u^2 du$

$= u - \frac{1}{3}u^3 + C$

$= \sin(x) - \frac{1}{3}\sin^3(x) + C$

3. $\int \sin^2(x)\cos^4(x)dx$

$= \int (1-\cos^2(x))\cos^4(x)dx$

$= \int \cos^4(x) - \cos^6(x)dx$

use $\cos^2(x) = \frac{1}{2}(\cos(2x)+1)$ a lot

$= \int \frac{1}{32}(2+\cos(2x) - 2\cos(4x) - \cos(6x))dx$

$= \frac{1}{32}\left[2x + \frac{1}{2}\sin(2x) - \frac{1}{2}\cos(4x) - \frac{1}{6}\cos(6x)\right] + C$

$= -\frac{\sin(6x)}{192} - \frac{\sin(4x)}{64} + \frac{\sin(2x)}{64} + \frac{x}{16} + C$

4. $\int \sec^3(x)\tan^5(x)dx$

$= \int \sec(x)\tan(x)\sec^2(x)\tan^4(x)dx$

note $u=\tan(x)$ $du=\sec^2(x)$ & we have an extra $\sec(x)$ we can't switch to tan

$= \int \underbrace{\sec(x)\tan(x)}_{du}\sec^2(x)(\sec^2(x)-1)^2dx$

$u=\sec(x)$ $\quad du=\sec(x)\tan(x)$

$= \int u^2(u^2-1)^2du$

$= \int u^6 - 2u^4 + u^2 du$

$= \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C$

$= \frac{1}{7}\sec^7(x) - \frac{2}{5}\sec^5(x) + \frac{1}{3}\sec^3(x) + C$

Exit Ticket Partial Fraction

Partial Fraction Decomposition:

denominator	partial fraction
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^n$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\ldots+\frac{A_n}{(ax+b)^n}$
ax^2+bx	$\frac{Ax+B}{ax^2+bx}$
$(ax^2+b)^n$	$\frac{A_1x+B_1}{ax^2+b}+\frac{A_2x+B_2}{(ax^2+b)^2}+\ldots+\frac{A_nx+B_n}{(ax^2+b)^n}$

Solve the following integrals using partial fractions:

1. $\displaystyle\int \frac{x^2+x+1}{(x+1)(x+4)^2}dx$

$$\frac{x^2+x+1}{(x+1)(x+4)^2} = \frac{A}{(x+1)} + \frac{B}{(x+4)} + \frac{C}{(x+4)^2}$$

$$x^2+x+1 = A(x+4)^2 + B(x+1)(x+4) + C(x+1)$$

$$x^2+x+1 = Ax^2+8Ax+16A+Bx^2+5Bx+4B+Cx+C$$

$x^2: 1=A+B$ | $A=1-B$

$x: 1=8A+5B+C$ | $1=8(1-B)+5B+C$

$k: 1=16A+4B+C$ | $1=16(1-B)+4B+C$

$A=1-B$ | $A=1-B$

$1=8-8B+5B+C$ | $-7=-3B+C$

$1=16-12B+C$ | $(-15=-12B+C)\cdot -1$

$A=1-B$ | $A=1-B$

$-7=-3B+C$ | $-7=-3B+C$

$15=12B-C$)+ | $8=9B$

$A=1-8/9$ | $A=1/9$

$-7=-3(8/9)+C$ | $C=-13/3$

$8/9=B$ | $B=8/9$

$$=\int \frac{1/9}{(x+1)} + \frac{8/9}{(x+4)} + \frac{-13/3}{(x+4)^2}dx$$

$$=\frac{1}{9}\ln|x+1| + \frac{8}{9}\ln|x+4| + \frac{13}{3}(x+4)^{-1} + C$$

2. $\displaystyle\int \frac{2x^4+x^3+4x^2+1}{x^3+x}dx$

$$\begin{array}{r} 2x+1 \\ x^3+x\,\overline{)\,2x^4+x^3+4x^2+1} \\ -(2x^4\quad -2x^2) \\ \hline 0+x^3+2x^2+1 \\ -(x^3+x) \\ \hline 0\quad 2x^2-x+1 \end{array}$$

$$=\int 2x+1+\frac{2x^2-x+1}{x(x^2+1)}dx$$

$$\frac{2x^2-x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$2x^2-x+1 = Ax^2+A+Bx^2+Cx$$

$x^2: 2=A+B$ | $2=1+B$ | $1=B$

$x: -1=C$ | $-1=C$ | $-1=C$

$k: 1=A$ | $1=A$ | $1=A$

$$=\int 2x+1+\frac{1}{x}+\frac{1x-1}{x^2+1}dx$$

$$=\int 2x+1+\frac{1}{x}+\frac{x}{x^2+1}-\frac{1}{x^2+1}dx$$

$$=x^2+x+\ln|x|+\frac{1}{2}\ln|x^2+1|-\arctan(x)+C$$