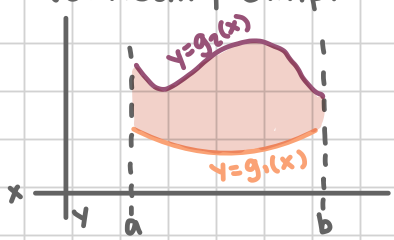
Double Integrals - General

Over General Regions

So far we have been working under the assumption the region we are working over is a rectangle, but this isn't always the case.

The integral over any region D can be described in two ways:

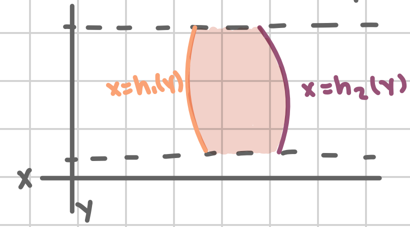
(i) Vertically Simple



$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

(ii) Horizontally Simple

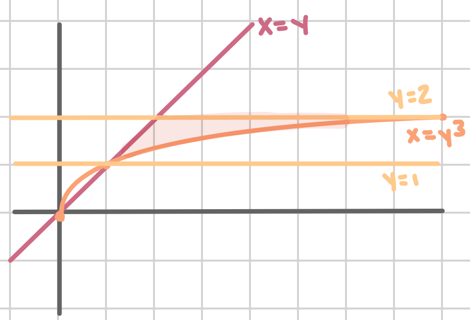


$D = \{(x,y) \mid h_1(y) \le x \le h_2(y),\ c \le y \le d\}$

$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

Examples:

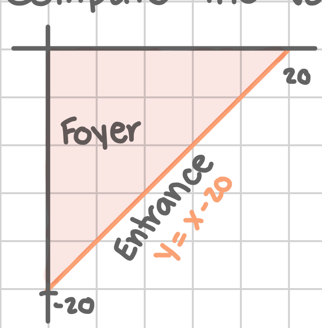
1. Compute $\iint_D e^{x/y}\,dA$ where $D = \{(x,y) \mid 1 \le y \le 2,\ y \le x \le y^3\}$



This horizontally simple i.e. every horizontal line I draw horizontal line in the shaded region is bounded on top by one function and on the bottom by another function.

$$\iint_D e^{x/y}\,dA = \int_1^2 \int_y^{y^3} e^{x/y}\,dx\,dy$$
$$= \int_1^2 \left[y e^{x/y}\right]_y^{y^3} dy$$
$$= \int_1^2 y e^{y^2} - y e^1\,dy$$
$$= \left[\tfrac{1}{2} e^{y^2} - \tfrac{1}{2} y^2 e^1\right]_1^2$$
$$= \tfrac{1}{2} e^4 - 2e^1$$

2. Compute the volume of the foyer in the house from last lecture.



The foyer is both vertically and horizontally simple

(i) vertically $D = \{(x,y) \mid 0 \le x \le 10,\ x-10 \le y \le 0\}$

$$\int_0^{10} \int_{x-10}^{0} 40 - 2x + 2y\,dy\,dx$$

(ii) horizontally $D = \{(x,y) \mid -10 \le y \le 0,\ 0 \le x \le y+10\}$

$$\int_{-10}^{0} \int_0^{y+10} 40 - 2x + 2y\,dx\,dy$$

(i) vertically $D = \{(x,y) \mid 0 \le x \le 10,\ x-10 \le y \le 0\}$

$$\int_0^{10}\int_{x-10}^{0} 40-2x+2y \; dy\,dx$$

integrate with respect to y

$$= \int_0^{10} 40y - 2xy + y^2 \Big|_{x-10}^{0} \, dx$$

plug into y

$$= \int_0^{10} 0 - (40(x-10) - 2x(x-10) + (x-10)^2 \, dx$$

$$= \int_0^{10} -(40x - 400 - 2x^2 + 20x + x^2 - 20x + 100) \, dx$$

$$= \int_0^{10} -(-x^2 + 40x - 300) \, dx$$

$$= \int_0^{10} x^2 - 40x + 300 \, dx$$

$$= \frac{1}{3}x^3 - 20x^2 + 300x \Big|_0^{10}$$

$$= \frac{1}{3}(10)^3 - 20(10)^2 + 300(10)$$

$$= \frac{4000}{3}$$

(ii) horizontally $D = \{(x,y) \mid -10 \le y \le 0,\ 0 \le x \le y+10\}$

$$\int_{-10}^{0}\int_0^{y+10} 40-2x+2y \; dx\,dy$$

$$= \int_{-10}^{0} 40x - x^2 + 2yx \Big|_0^{y+10} \; dy$$

$$= \int_{-10}^{0} 40(y+10) - (y+10)^2 + 2y(y+10) \; dy$$

$$= \int_{-10}^{0} 40y + 400 - y^2 - 20y - 100 + 2y^2 + 20y \; dy$$

$$= \int_{-10}^{0} y^2 + 40y + 300 \; dy$$

$$= \frac{1}{3}y^3 + 20y^2 + 300y \Big|_{-10}^{0}$$

$$= 0 - \left(\frac{1}{3}(-10)^3 + 20(-10)^2 + 300(-10)\right)$$

$$= \frac{4000}{3}$$

Exit Ticket Numerical Integration

Numerical Integration We can estimate the integral $\int_a^b f(x)dx$ using the following formulas,

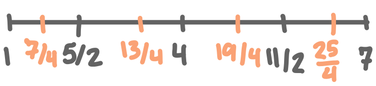
1. **midpoint:** $\int_a^b f(x)dx \approx \Delta x\left[f(x_1^*) + f(x_2^*) + ... + f(x_n^*)\right]$
2. **trapezoid:** $\int_a^b f(x)dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + ... + 2f(x_{n-1} + f(x_n)\right]$
3. **simpson's:** $\int_a^b f(x)dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2)... + 2f(x_{n-1}) + 4f(x_{n-1}) + f(x_n)\right]$

where n is the number is subintervals and $\Delta x = \frac{b-a}{n}$

Estimate the following integrals using each of the rules above: (with n=4)

1. $\int_1^7 \frac{1}{x^3+1}dx$

$\Delta x = \frac{7-1}{4} = \frac{6}{4} = \frac{3}{2}$



midpoint:

$= \frac{3}{2}\left[\frac{1}{(7/4)^3+1} + \frac{1}{(13/4)^3+1} + \frac{1}{(19/4)^3+1} + \frac{1}{(25/4)^3+1}\right]$

trapezoid:

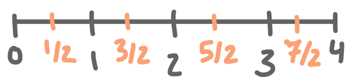
$= \frac{3}{4}\left[\frac{1}{(1)^3+1} + 2\cdot\frac{1}{(5/2)^3+1} + 2\cdot\frac{1}{(4)^3+1} + 2\cdot\frac{1}{(11/2)^3+1} + \frac{1}{(7)^3+1}\right]$

simpson's:

$= \frac{1}{2}\left[\frac{1}{(1)^3+1} + 4\cdot\frac{1}{(5/2)^3+1} + 2\cdot\frac{1}{(4)^3+1} + 4\cdot\frac{1}{(11/2)^3+1} + \frac{1}{(7)^3+1}\right]$

2. $\int_0^4 \cos(1+\sqrt{x})dx$

$\Delta x = \frac{4-0}{4} = 1$



midpoint:

$= 1\left[\cos(1+\sqrt{\tfrac{1}{2}}) + \cos(1+\sqrt{\tfrac{3}{2}}) + \cos(1+\sqrt{\tfrac{5}{2}}) + \cos(1+\sqrt{\tfrac{7}{2}})\right]$

trapezoid:

$= \frac{1}{2}\left[\cos(1) + 2\cos(2) + 2\cos(1+\sqrt{2}) + 2\cos(1+\sqrt{3}) + \cos(3)\right]$

simpson's:

$= \frac{1}{3}\left[\cos(1) + 4\cos(2) + 2\cos(1+\sqrt{2}) + 4\cos(1+\sqrt{3}) + \cos(3)\right]$

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