Introduction to Polar Coordinates

In Polar Coordinates

So far the region D could either be described by Cartesian coordinates or by functions of cartesian coordinates. Sometimes a region is better described in terms of polar coordinates like disks, ring, or portions of disks and rings. For instance if D is the disk of radius 2 then D can be described by $-2 \le x \le 2$ and $-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$ OR $0 \le \theta \le 2\pi$ and $0 \le r \le 2$, which set of integrals looks easier to solve:

$$\iint_D f(x,y)\,dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx = \int_0^{2\pi}\int_0^2 f(r\cos\theta, r\sin\theta)\cdot r\cdot dr\,d\theta.$$

To convert from rectangular to polar:

$r^2 = x^2 + y^2$

$\alpha = \tan^{-1}\left(\frac{y}{x}\right) + \pi$ (sometimes α is not enough)

To convert polar to rectangular:

$x = r\cos\theta$

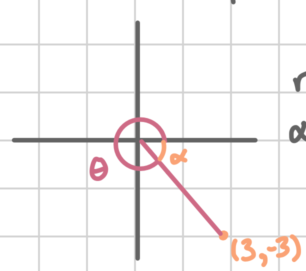
$y = r\sin\theta$

$dx\,dy = r\cdot dr\,d\theta$

Examples:

1. Convert each of the following points into the given coordinate system.

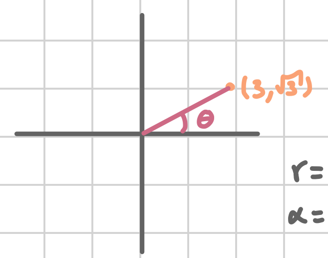
(a) $(3,-3)$ into polar



$r = \sqrt{(3)^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$

$\alpha = \tan^{-1}\left(\frac{-3}{3}\right) = -\frac{\pi}{4}$

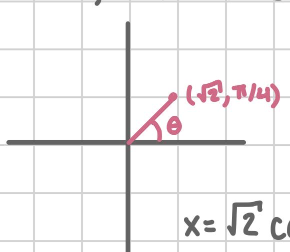
(b) $(3, \sqrt{3})$ into polar



$r = \sqrt{(3)^2 + (\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3}$

$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \pi/6$

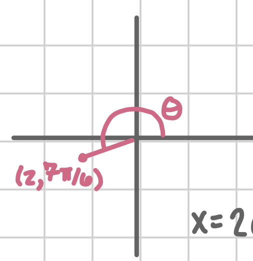
(c) $(\sqrt{2}, \pi/4)$ into rectangular



$x = \sqrt{2}\cos(\pi/4) = \sqrt{2}\left(\frac{\sqrt{2}}{2}\right) = 1$

$y = \sqrt{2}\sin(\pi/4) = \sqrt{2}\left(\frac{\sqrt{2}}{2}\right) = 1$

(d) $(2, 7\pi/6)$ into rectangular.



$x = 2\cos(7\pi/6) = 2(-\sqrt{3}/2) = -\sqrt{3}$

$y = 2\sin(7\pi/6) = 2(-1/2) = -1$

2. Convert each of the following equations into the given coordinate system.

(a) $2x - 5x^3 = 1 + xy$ into polar

$2(r\cos\theta) - 5(r\cos\theta)^3 = 1 + (r\cos\theta)(r\sin\theta)$

$2r\cos\theta - 5r^3\cos^3\theta = 1 + r^2\cos\theta\sin\theta$

(b) $r = -8\cos\theta$ into cartesian

$r^2 = -8r\cos\theta$

$x^2 + y^2 = -8x$

3. Find the volume of the upper portion of the hobbit house with height $h(x,y) = 40 - 2x + 2y$.

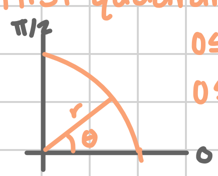
$$\iint_D h(x,y)\,dA = \int_0^{\pi}\int_0^{10} (40 - 2r\cos\theta + 2r\sin\theta)\cdot r\,dr\,d\theta$$
$$= \int_0^{\pi}\int_0^{10} 40r - 2r^2\cos\theta + 2r^2\sin\theta\,dr\,d\theta$$
$$= \int_0^{\pi} 20r^2 - \tfrac{2}{3}r^3\cos\theta + \tfrac{2}{3}r^3\sin\theta\,\Big|_0^{10}\,d\theta$$
$$= \int_0^{\pi} 20(10)^2 - \tfrac{2}{3}(10)^3\cos\theta + \tfrac{2}{3}(10)^3\sin\theta\,d\theta$$
$$= \int_0^{\pi} 2(10)^3 - \tfrac{2}{3}(10)^3\cos\theta + \tfrac{2}{3}(10)^3\sin\theta\,d\theta$$
$$= \tfrac{2}{3}(10)^3\int_0^{\pi} 3 - \cos\theta + \sin\theta\,d\theta$$
$$= \tfrac{2}{3}(10)^3\left[3\theta - \sin\theta - \cos\theta\right]_0^{\pi}$$
$$= \tfrac{2}{3}(10)^3\left[3\pi - 0 - (-1) - (0 - 0 - (1))\right]$$
$$= \tfrac{2}{3}(10)^3\left[3\pi\right]$$
$$= 2(10)^3\pi$$

$D: \{(x,y) \mid -10 \le x \le 10,\ 0 \le y \le \sqrt{10^2 - x^2}\}$

OR $\{(r,\theta) \mid 0 \le r \le 10,\ 0 \le \theta \le \pi\}$

4. The density of a quarter of a disc of radius 2m centered at the origin sitting in the first quadrant is given by the function $f(x,y) = 2x^2 + y^2$. Find the total mass of the quarter disc.

first quadrant:



$0 \le x \le 2$

$0 \le \theta \le \frac{\pi}{2}$

$2x^2 + y^2$

$= x^2 + x^2 + y^2$

$= (r\cos\theta)^2 + r^2$

$dA = r \cdot dr\,d\theta$

$$\text{Mass} = \iint_R \rho(x,y)\,dA$$
$$= \int_0^2\int_0^{\sqrt{r^2 - x^2}} 2x^2 + y^2\,dy\,dx$$
$$= \int_0^2\int_0^{\pi/2} (r^2\cos^2\theta + r^2)\,r\,d\theta\,dr$$
$$= \int_0^2\int_0^{\pi/2} r^3\cos^2\theta + r^3\,d\theta\,dr$$
$$= \int_0^2\int_0^{\pi/2} r^3\left(\tfrac{1}{2}(1 + \cos(2\theta))\right) + r^3\,d\theta\,dr$$
$$= \int_0^2 \left[\tfrac{1}{2}r^3\left(\theta + \tfrac{1}{2}\sin(2\theta)\right) + r^3\theta\right]_0^{\pi/2} dr$$
$$= \int_0^2 \tfrac{\pi}{4}r^3 + \tfrac{1}{4}r^3\cancelto{0}{\sin(\pi)} + \tfrac{\pi}{2}r^3 - \left(0 + \tfrac{1}{4}r^3\cancelto{0}{\sin(0)} + 0\right)dr$$
$$= \int_0^2 \tfrac{3}{4}r^3\,dr$$
$$= \left[\tfrac{3}{16}r^4\right]_0^2$$
$$= 3 - \tfrac{3}{16} = \tfrac{45}{16}$$

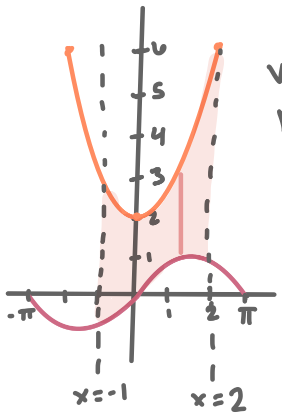
Exit Ticket Double Integrals

Double Integrals The integral over a horizontally simple region $D = \{(x, y) \mid h_1(y) \le x \le h_2(y),\ c \le y \le d\}$ is

$$\int\int_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Find the integral $\int\int_D 2x\,dA$ for the regions bounded by the following functions:

1. $y = x^2 + 2,\ y = \sin(x),\ x = -1,\ x = 2$



vertically simple:

$D = \{(x,y) \mid -1 \le x \le 2,\ \sin(x) \le y \le x^2+2\}$

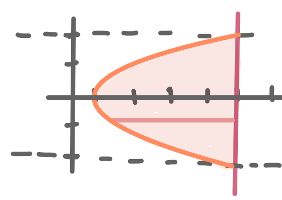
$\int_{-1}^{2}\int_{\sin(x)}^{x^2+2} 2x\,dy\,dx$

$= \int_{-1}^{2} 2xy \Big|_{\sin(x)}^{x^2+2} dx$

$= \int_{-1}^{2} 2x^3 + 4x - \underbrace{2x\sin(x)}_{\text{by parts}}\,dx$

$= \frac{1}{2}x^4 + 2x^2 + 2x\cos(x) - 2\sin(x) \Big|_{-1}^{2}$

2. $x = y^2 + 1,\ x = 5,\ y = \pm 2$



vertical fails because $g_1(x) = g_2(x)$

horizontally simple

$D = \{(x,y) \mid -2 \le y \le 2,\ y^2+1 \le x \le 5\}$

$\int_{-2}^{2}\int_{y^2+1}^{5} 2x\,dx\,dy$

$= \int_{-2}^{2} x^2 \Big|_{y^2+1}^{5} dy$

$= \int_{-2}^{2} 25 - (y^4 + 2y^2 + 1)\,dy$

$= 25y - \frac{1}{5}y^5 - \frac{2}{3}y^3 - y \Big|_{-2}^{2}$

$= \frac{1088}{15}$

3. $y = \frac{1}{x+2},\ y = (x+2)^2,\ x = -1,\ x = 1$

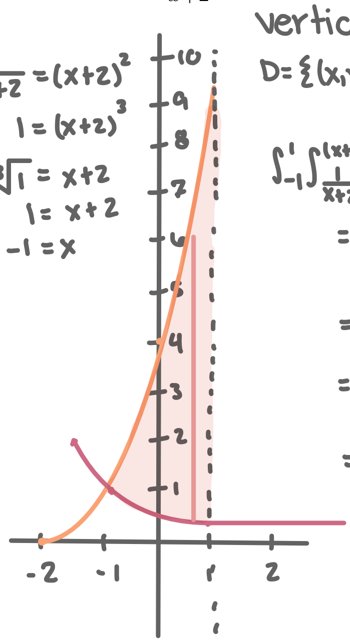
$\frac{1}{x+2} = (x+2)^2$

$1 = (x+2)^3$

$\sqrt[3]{1} = x+2$

$1 = x+2$

$-1 = x$



vertically simple

$D = \{(x,y) \mid -1 \le x \le 1,\ \frac{1}{x+2} \le y \le (x+2)^2\}$

$\int_{-1}^{1}\int_{\frac{1}{x+2}}^{(x+2)^2} 2x\,dy\,dx$

$= \int_{-1}^{1} 2xy \Big|_{\frac{1}{x+2}}^{(x+2)^2} dx$

$= \int_{-1}^{1} 2x(x+2)^2 - \frac{2x}{x+2}\,dx$ partial fractions

$= \int_{-1}^{1} \frac{2x(x+2)^3 - 2x}{(x+2)}\,dx$

$= 2\int_{-1}^{1} \frac{x((x+2)^3 - 1)}{(x+2)}\,dx$

$u = x+2 \quad du = 1dx$

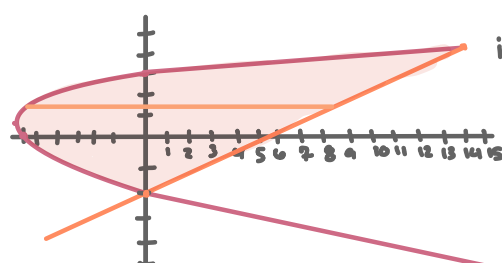
$= 2\int_{1}^{3} \frac{(u-2)(u^3-1)}{u}\,du$

$= 2\int_{1}^{3} \frac{u^4 - 2u^3 - u + 2}{u}\,du$

$= 2\int_{1}^{3} u^3 - 2u^2 - 1 + \frac{2}{u}\,du$

4. $x = y^2 - y - 6,\ x = 2y + 4$

$x = (y-3)(y+2)$



intercepts:

$y^2 - y - 6 = 2y + 4$

$y - 3y - 10 = 0$

$(y-5)(y+2) = 0$

$y = -2, 5$

horizontally simple

$D = \{(x,y) \mid -2 \le y \le 5,\ y^2 - y - 6 \le x \le 2y+4\}$

$\int_{-2}^{5}\int_{y^2-y-6}^{2y+4} 2x\,dx\,dy$

$= \int_{-2}^{5} x^2 \Big|_{y^2-y-6}^{2y+4} dy$

$= \int_{-2}^{5} (2y+4)^2 - (y^2-y-6)^2\,dy$