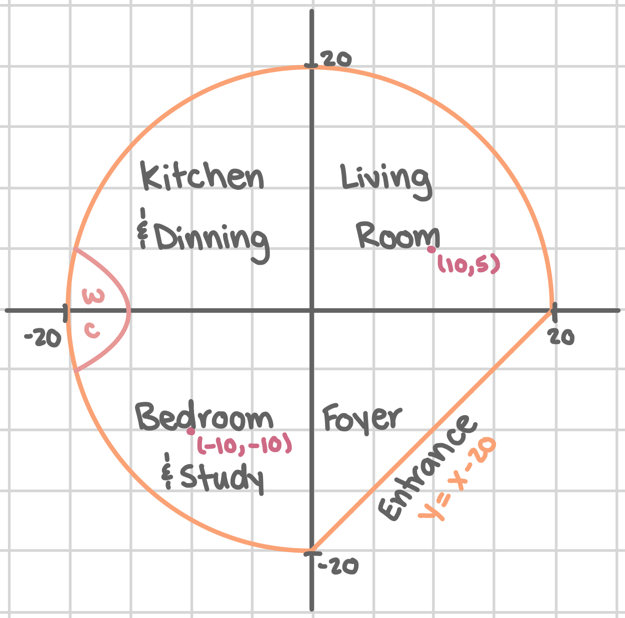
Week 08: March 9th, 2023

Section 14.1, 15.1, & 15.2: Two Variable Functions & Double Integrals

So far we have exclusively looked at functions of the form $y=f(x)$ and $x=h(y)$, but not all curves or equations follow this form. For example, a circle. The equation of a circle centered at $(0,0)$ with radius r is given by $x^2+y^2=r$. This equation can not be transformed into one of the forms mentioned above. We can solve for x or y but are left with $y=\pm\sqrt{r^2-x^2}$ or $x=\pm\sqrt{r^2-y^2}$ but these are technically two functions each, the top and bottom hemispheres or the left and right hemisphere.

Let us get familiar with these two variable functions with a visual example. Let's say we have a house with a circular floor plan with a cut-off for the entrance. The height at any point in this house can be given by the two variable function $f(x,y)=14-\frac{1}{100}(x^2+y^2)$. Using the picture below answer the questions to the side.



(a) What region of the house are you in at $(10,5)$ and $(-10,-10)$?

$(10,5)$ is in the living room and $(-10,-10)$ is in the living room & study

(b) What is the height at $(10,5)$ and $(-10,-10)$?

$f(10,5)=14-\frac{1}{100}((10)^2+(5)^2)=14-\frac{1}{100}(125)=14-1.25=12.75$

$f(-10,-10)=14-\frac{1}{100}((-10)^2+(-10)^2)=14-\frac{1}{100}(200)=14-2=12$

(c) Where is the highest point? What is the height around the circular perimeter?

Highest point happens when we subtract 0 from 14, i.e. $(0,0)$.

Around circular edge $f(x,y)=14-\frac{1}{100}(x^2+y^2)=14-\frac{1}{100}(r^2)=14-\frac{1}{100}((20)^2)=14-4=10$

(d) Write down the height function along the entrance in terms of x with a range.

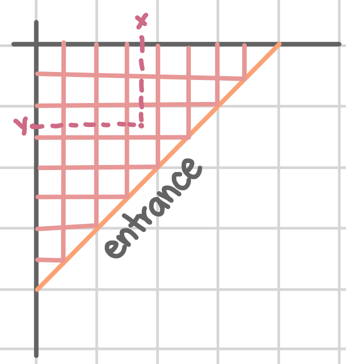
Using the equation of the line $y=x-20$, $f(x,y)=f(x,x-20)=14-\frac{1}{100}(x^2+(x-20)^2)$

(e) Parameterize (or describe) the foyer region in terms of x and y.

x ranges from 0 to 20 and y ranges (in x terms) from $x-20$ to 0

(f) Use the ideas of Riemann sums to find the internal volume of the house enclosed by the foyer.

Volume enclosed by the surface $f(x,y)=f(x,x-20)=14-\frac{1}{100}(x^2-y^2)$ over the region $D=\{(x,y)\mid 0\leq x\leq 20,\ x-20\leq y\leq 0\}$.



Divide the region D up into small rectangular patches with evenly spaced horizontal and vertical lines. We describe each square by its center (x,y) with width Δx and length Δy so it has area $A=\Delta x\cdot\Delta y$. Since this square is infinitely small, we can assume the height to be $f(x,y)$. Thus $\Delta V=\Delta x\cdot\Delta y\cdot f(x,y)=(14-\frac{1}{100}(x^2+y^2))\cdot\Delta x\cdot\Delta y$. Just like before we can sum over the variables to get $V=\lim_{n,m\to\infty}\sum^{n}\sum^{m} 14-\frac{1}{100}(x^2+y^2)\,\Delta y\,\Delta x$.

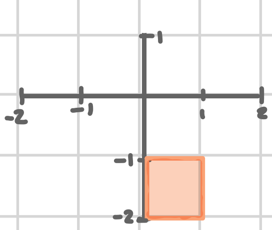
Double Integrals

Rectangular Coordinates

The double integration of $f(x,y)$ over the rectangle $R=\{(x,y)\mid a\leq x\leq b,\ c\leq y\leq d\}$ is $\iint_R f(x,y)\,dA=\int_c^d\int_a^b f(x,y)\,dx\,dy$ (if it exists).

example. Compute $\iint_R x^2y^2+\cos(\pi x)+\sin(\pi y)\,dA$ over $R=[-2,-1]\times[0,1]$.

$\iint_R x^2y^2+\cos(\pi x)+\sin(\pi y)\,dA=\int_0^1\int_{-2}^{-1} x^2y^2+\cos(\pi x)+\sin(\pi y)\,dx\,dy$



$=\int_0^1\left[\frac{1}{3}x^3y^2+\frac{1}{\pi}\sin(\pi x)+x\sin(\pi y)\right]_{-2}^{-1}dy$ sin(πy) is a constant with respect to x

$=\int_0^1\frac{7}{3}y^2+\sin(\pi y)\,dy$

$=\left[\frac{7}{9}y^3-\frac{1}{\pi}\cos(\pi y)\right]_0^1$

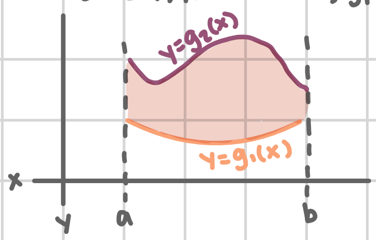
$=\frac{7}{9}+\frac{2}{\pi}$

Over General Regions

So far we have been working under the assumption the region we are working over is a rectangle, but this isn't always the case. The integral over any region D can be described in two ways:

(i) Vertically Simple

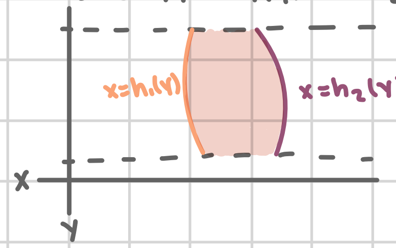
$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$



$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

(ii) Horizontally Simple

$D = \{(x,y) \mid h_1(y) \le x \le h_2(y),\ c \le y \le d\}$



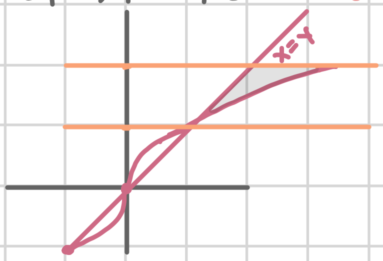
$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

example. Compute $\iint_D e^{x/y}\,dA$ where $D = \{(x,y) \mid 1 \le y \le 2,\ y \le x \le y^3\}$ (this is case (ii))

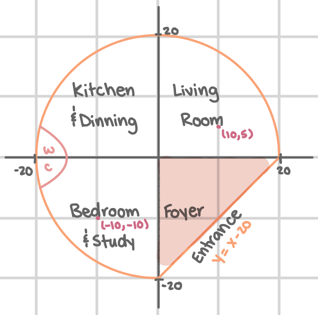
$\iint_D e^{x/y}\,dA = \int_1^2 \int_y^{y^3} e^{x/y}\,dx\,dy$

$= \int_1^2 \left[y e^{x/y} \right]_y^{y^3} dy \quad = \int_1^2 y e^{y^2} - y e^1\,dy$

$= \left[\frac{1}{2} e^{y^2} - \frac{1}{2} y^2 e^1 \right]_1^2 \quad = \frac{1}{2} e^4 - 2e^1$



example. Compute the volume of the foyer in the house example above.



The foyer is both vertically and horizontally simple so we can do it two ways.

(i) vertically $D = \{(x,y) \mid 0 \le x \le 20,\ x-20 \le y \le 0\}$

$\int_0^{20} \int_{x-20}^{0} 14 - \frac{1}{100}(x^2+y^2)\,dy\,dx$

$= \int_0^{20} \int_{x-20}^{0} 14 - \frac{1}{100}x^2 - \frac{1}{100}y^2\,dy\,dx$

$= \int_0^{20} 14y - \frac{1}{100}x^2 y - \frac{1}{100}\cdot\frac{1}{3}y^3 \Big]_{x-20}^{0}\,dx$

$= \int_0^{20} 14(x-20) - \frac{1}{100}x^2(x-20) - \frac{1}{300}(x-20)^3\,dx$

$= \int_0^{20} 14x - 280 - \frac{1}{100}x^3 - \frac{1}{5}x^2 - \frac{1}{300}x^3 - \frac{1}{5}x^2 - 4x + \frac{80}{3}\,dx$

$= \int_0^{20} -\frac{1}{75}x^3 - \frac{2}{5}x^2 + 10x + \frac{760}{3}\,dx$

$= -\frac{1}{300}x^4 - \frac{2}{15}x^3 + 5x^2 + \frac{760}{3}x \Big]_0^{20}$

$= -\frac{1}{300}(20)^4 - \frac{2}{15}(20)^3 + 5(20)^2 + \frac{760}{3}(20)$

$= \frac{7600}{3}$

(ii) horizontally $D = \{(x,y) \mid -20 \le y \le 0,\ 0 \le x \le y+20\}$

$\int_{-20}^{0} \int_0^{y+20} 14 - \frac{1}{100}(x^2+y^2)\,dx\,dy$

$= \int_{-20}^{0} \int_0^{y+20} 14 - \frac{1}{100}x^2 - \frac{1}{100}y^2\,dx\,dy$

$= \int_{-20}^{0} 14x - \frac{1}{300}x^3 - \frac{1}{300}y^2 x \Big]_0^{y+20}\,dy$

$= \int_{-20}^{0} 14(y+20) - \frac{1}{300}(y+20)^3 - \frac{1}{300}y^2(y+20)\,dy$

$= \int_{-20}^{0} -\frac{1}{75}y^3 - \frac{2}{5}y^2 + 10y + \frac{760}{3}$

$= \frac{-1}{300}y^4 - \frac{3}{15}y^3 + 5y^2 + \frac{760}{3}y \Big]_{-20}^{0}$

$= 0 - \left(\frac{-1}{300}(-20)^4 - \frac{3}{15}(-20)^3 + 5(-20)^2 + \frac{760}{3}(-20)\right)$

$= \frac{7600}{3}$

In Polar Coordinates

So far the region D could either be described by Cartesian coordinates or by functions of cartesian coordinates. Sometimes a region is beter described in terms of polar coordinates like disks, ring, or portions of disks and rings. For instance if D is the disk of radius 2 then D can be described by $-2 \le x \le 2$ and $-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$ OR $0 \le \theta \le 2\pi$ and $0 \le r \le 2$, which set of integrals looks easier to solve:

$\iint_D f(x,y)\,dA = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx = \int_0^{2\pi} \int_0^2 f(r\cos\theta, r\sin\theta)\cdot r \cdot dr\,d\theta.$

To convert from rectangular to polar:

$r^2 = x^2 + y^2$

$\alpha = \tan^{-1}\left(\frac{y}{x}\right) + \pi$ (sometimes α is not enough)

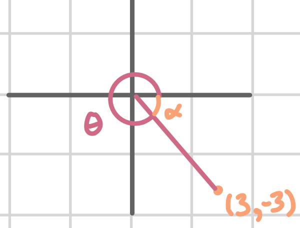
To convert polar to rectangular:

$x = r\cos\theta$

$y = r\sin\theta$

example. Convert each of the following points into the given coordinate system.

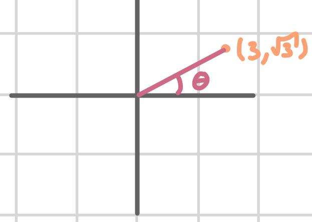
(a) $(3,-3)$ into polar



$r = \sqrt{(3)^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$

$\alpha = \tan^{-1}\left(\frac{-3}{3}\right) = -\frac{\pi}{4}$

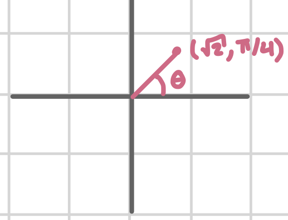
(b) $(3, \sqrt{3})$ into polar



$r = \sqrt{(3)^2 + (\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3}$

$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \pi/6$

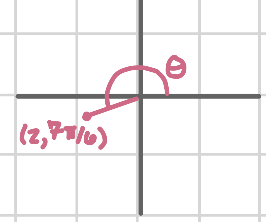
(c) $(\sqrt{2}, \pi/4)$ into rectangular



$x = \sqrt{2}\cos(\pi/4) = \sqrt{2}\left(\frac{\sqrt{2}}{2}\right) = 1$

$y = \sqrt{2}\sin(\pi/4) = \sqrt{2}\left(\frac{\sqrt{2}}{2}\right) = 1$

(d) $(2, 7\pi/6)$ into rectangular.



$x = 2\cos(7\pi/6) = 2(-\sqrt{3}/2) = -\sqrt{3}$

$y = 2\sin(7\pi/6) = 2(-1/2) = -1$

example. Convert each of the following equations into the given coordinate system.

(a) $2x - 5x^3 = 1 + xy$ into polar

$2(r\cos\theta) - 5(r\cos\theta)^3 = 1 + (r\cos\theta)(r\sin\theta)$

$2r\cos\theta - 5r^3\cos^3\theta = 1 + r^2\cos\theta\sin\theta$

(b) $r = -8\cos\theta$ into cartesian

$r^2 = -8r\cos\theta$

$x^2 + y^2 = -8x$

example. The density of a quarter of a disc of radius 2m centered at the origin sitting in the first quadrant is given by the function $f(x,y) = 2x^2 + y^2$. Find the total mass of the quarter disc.

$$\text{Mass} = \iint_R \rho(x,y)\,dA$$

$$= \int_0^2 \int_0^{\sqrt{r^2 - x^2}} 2x^2 + y^2 \, dy\,dx$$

$2x^2 + y^2$

$$= \int_0^2 \int_0^{\pi/2} (r^2\cos^2\theta + r^2)\, r\, d\theta\, dr$$

$x^2 + x^2 + y^2$

$$= \int_0^2 \int_0^{\pi/2} r^3\cos^2\theta + r^3 \, d\theta\, dr$$

$(r\cos\theta)^2 + r^2$

$$= \int_0^2 \int_0^{\pi/2} r^3\left(\tfrac{1}{2}(1 + \cos(2\theta))\right) + r^3 \, d\theta\, dr$$

$$= \int_0^2 \left[\tfrac{1}{2} r^3 \left(\theta + \tfrac{1}{2}\sin(2\theta)\right) + r^3\theta \right]_0^{\pi/2} dr$$

$$= \int_0^2 \tfrac{\pi}{4} r^3 + \tfrac{1}{4} r^3 \cancel{\sin(\pi)}^{0} + \tfrac{\pi}{2} r^3 - \left(0 + \tfrac{1}{4} r^3 \cancel{\sin(0)}^{0} + 0\right) dr$$

$$= \int_0^2 \tfrac{3}{4} r^3 \, dr$$

$$= \left[\tfrac{3}{16} r^4 \right]_0^2$$

$$= 3 - \tfrac{3}{16}$$

$$= \tfrac{45}{16}$$

example. Compute $\iint_D 2xy\,dA$ where D is the region between the circle of radius 2 centered at the origin and the circle of radius 5 centered at the origin that lies in the first quadrant.

D: $2 \le r \le 5$ and $0 \le \theta \le \pi/2$

$$\iint_D 2xy\,dA = \int_0^{\pi/2} \int_2^5 2(r\cos\theta)(r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \int_2^5 r^3 \sin(2\theta)\, dr\, d\theta$$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$$= \int_0^{\pi/2} \left[\tfrac{1}{4} r^4 \sin(2\theta) \right]_2^5 d\theta = \int_0^{\pi/2} \tfrac{609}{4} \sin(2\theta)\, d\theta$$

$$= \left[-\tfrac{609}{8} \cos(2\theta) \right]_0^{\pi/2} = \tfrac{609}{4}$$