Separable Differential Equations

Differential Equations

A differential equation is an equation that involves an unknown function and its first or higher derivatives.

examples.

$$\frac{dy}{dx} = 1 - 6e^{2x}$$

$$\frac{dy}{dt} + \frac{1}{t+30} \cdot y = 4$$

first order differential equations

$$\frac{d^2y}{d^2x} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

second order (homogeneous) diff. eq.

$$\frac{dy}{dx} = F(x,y)$$

general first order differential equations

Separation of Variables

Separable differential equation: $\frac{dy}{dx} = p(x)q(y)$

examples.

$\frac{dy}{dx} = 1-6e^{2x}$ $\quad p(x) = 1-6e^{2x}$; $q(y) = 1$

$y'(x) = 3x^2y$ $\quad p(x) = 3x^2$; $q(y) = y$

How to solve? Method of separation.

$$\frac{dy}{dx} = p(x)q(y) \iff y'(x) = p(x)q(y)$$

$$\iff \frac{1}{q(y)} y'(x) = p(x)$$

$$\iff \int \frac{1}{q(y)} y'(x)\,dx = \int p(x)\,dx$$

$$\iff \int \frac{1}{q(y)}\,dy = \int p(x)\,dx$$

Simplified explanation:

$$\frac{dy}{dx} = p(x)q(y) \iff \frac{1}{q(y)}\,dy = p(x)\,dx$$

$$\iff \int \frac{1}{q(y)}\,dy = \int p(x)\,dx$$

Examples:

1. Solve the differential equation $\frac{dy}{dx} = 1 - 6e^{2x}$ for
   (i) all solutions and
   (ii) the solution satisfying $y(0)=5$

(i) $\frac{dy}{dx} = 1 - 6e^{2x}$

$\int 1\,dy = \int 1 - 6e^{2x}\,dx$

$y = x - 3e^{2x} + c$

This is a more complex phrasing of the initial value problem: given the slope $\frac{dy}{dx}$ & point (x_0, y_0) find the function $y(x)$.

(ii) @ $y(0) = 5$

$0 - 3e^0 + c = 5$

$-3 + c = 5$

$c = 8$

$y = x + 3e^{2x} + 8$

2. Solve for the general solution of $y'(x) = 3x^2 y$. Find the particular solution such that $y(0) = -2$.

$\frac{dy}{dx} = 3x^2 y$

$\frac{1}{y}dy = 3x^2\,dx$

$\int \frac{1}{y}dy = \int 3x^2\,dx$

$\ln|y| + c_1 = x^3 + c_2$

$\ln|y| = x^3 + (c_2 - c_1)$

$e^{\ln|y|} = e^{x^3 + (c_2 - c_1)}$

$y = e^{c_2 - c_1} \cdot e^{x^3}$

$y(x) = Ae^{x^3}$

@ $y(0) = -2$

$Ae^0 = -2$

$A = -2$

$y(x) = -2e^{x^3}$

3. Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings. A roast turkey is taken from an oven when its temperature is 185°F and is placed on a table in a room where the temperature is 75°F. Temperature of the turkey falls to 150°F after half an hour. Apply Newton's Law of Cooling to find the temperature of the turkey after 45 minutes.

Newton's Law of Coding: $y' = -k(y - T_0)$; $y(t)$ = temp. of object at time t, k = cooling constant, T_0 = ambient temp.

What we know: $T_0 = 75°F$, $y(0) = 185°F$, $y(30) = 150°F$

Need to find: k, $y(t)$, $y(45)$

general formula:

$$\frac{dy}{dt} = -k(y-75)$$

$$\frac{1}{y-75}\,dy = -k\,dt$$

$$\int \frac{1}{y-75}\,dy = \int -k\,dt$$

$$\ln|y-75| = -kt + c$$

$$y - 75 = e^{-kt+c}$$

$$y = e^c e^{-kt} + 75$$

$$y = Ae^{-kt} + 75$$

initial conditions:

@ $y(0) = 185$

$$185 = Ae^0 + 75$$

$$185 = A + 75$$

$$110 = A$$

@ $y(30) = 150$

$$110e^{-30k} + 75 = 150$$

$$110e^{-30k} = 75$$

$$e^{-30k} = 75/110$$

$$-30k = \ln\left(\frac{75}{110}\right)$$

$$k = -\frac{1}{30}\ln\left(\frac{75}{110}\right)$$

specified time:

$$y(t) = 110e^{-\frac{1}{30}\ln\left(\frac{75}{110}\right)t} + 75$$

$$y(45) = 110e^{-\frac{1}{30}\ln\left(\frac{75}{110}\right)45} + 75$$

$$= 110e^{-\frac{3}{2}\ln\left(\frac{75}{110}\right)} + 75$$

simplified formula:

$$y(t) = 110e^{-\frac{1}{30}\ln\left(\frac{75}{110}\right)t} + 75$$

$$= 110e^{\ln\left(\left(\frac{15}{22}\right)^{-\frac{1}{30}t}\right)} + 75$$

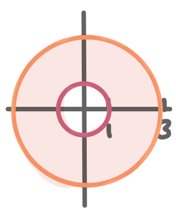
$$= 110\left(\frac{15}{22}\right)^{-1/30t} + 75$$

Exit Ticket Polar Integrals

> **Polar Integrals** The integral over region $D = \{(r,\theta) \mid a \le r \le b,\ c \le \theta \le d\,\}$ is
>
> $$\int\int_D f(r,\theta)\,dA = \int_c^d \int_a^b f(r,\theta)\cdot r\cdot dr d\theta$$

Find the integral $\int\int_D 2x^2 + y^2\,dA$ for the regions bounded by the following:

1. the circles of radius 1 and 3 centered at the origin



$1 \le r \le 3$

$0 \le \theta \le 2\pi$

$2x^2+y^2$

$= x^2+x^2+y^2$

$= r^2\cos^2\theta + r^2$

$\int_0^{2\pi}\int_1^3 (r^2\cos^2\theta + r^2)\cdot r\,dr\,d\theta$

$= \int_0^{2\pi}\int_1^3 r^3\cos^2\theta + r^3\,dr\,d\theta$

$= \int_0^{2\pi} \frac{1}{4}r^4\cos^2\theta + \frac{1}{4}r^4 \Big|_1^3 d\theta$

$= \int_0^{2\pi} \frac{1}{4}r^4(\cos^2\theta+1)\Big|_1^3 d\theta$

$= \int_0^{2\pi} \frac{1}{4}(\overset{81}{3^4}-1^4)(\cos^2\theta+1)\,d\theta$

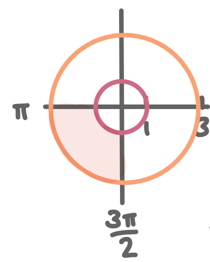
$= 20\int_0^{2\pi} \cos^2\theta + 1\,d\theta$

$= 20\int_0^{2\pi} \left(\frac{1}{2}\cos(2\theta)+\frac{1}{2}\right)+1\,d\theta$

$= 20\left[\frac{1}{4}\sin(2\theta)+\frac{3}{2}\theta\right]_0^{2\pi}$

$= 20[3\pi] = 60\pi$

2. the circles of radius 1 and 3 centered at the origin contained in the third quadrant



$1 \le r \le 3$

$\pi \le \theta \le \frac{3\pi}{2}$

$\int_\pi^{\frac{3\pi}{2}}\int_1^3 (r^2\cos^2\theta + r^2)\cdot r\,dr\,d\theta$

$= \int_\pi^{\frac{3\pi}{2}}\int_1^3 r^3\cos^2\theta + r^3\,dr\,d\theta$

$= \int_\pi^{\frac{3\pi}{2}} \frac{1}{4}r^4\cos^2\theta + \frac{1}{4}r^4 \Big|_1^3 d\theta$

$= \int_\pi^{\frac{3\pi}{2}} \frac{1}{4}r^4(\cos^2\theta+1)\Big|_1^3 d\theta$

$= \int_\pi^{\frac{3\pi}{2}} \frac{1}{4}(\overset{81}{3^4}-1^4)(\cos^2\theta+1)\,d\theta$

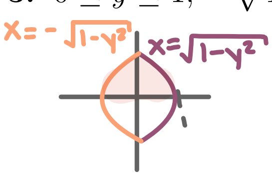
$= 20\int_\pi^{\frac{3\pi}{2}} \cos^2\theta + 1\,d\theta$

$= 20\int_\pi^{\frac{3\pi}{2}} \left(\frac{1}{2}\cos(2\theta)+\frac{1}{2}\right)+1\,d\theta$

$= 20\left[\frac{1}{4}\sin(2\theta)+\frac{3}{2}\theta\right]_\pi^{3\pi/2}$

$= 20\left[\frac{9\pi}{4}-\frac{3}{2}\pi\right]$

3. $0 \le y \le 1;\ -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$



$x = \sqrt{1-y^2}$

$x^2 = 1-y^2$

$x^2+y^2 = 1$

$0 \le r \le 1$

$0 \le \theta \le \pi$

$\int_0^{\pi}\int_0^1 (r^2\cos^2\theta + r^2)\cdot r\,dr\,d\theta$

$= \int_0^{\pi}\int_0^1 r^3(\cos^2\theta+1)\,dr\,d\theta$

$= \int_0^{\pi} \frac{1}{4}r^4\Big|_0^1(\cos^2\theta+1)\,d\theta$

$= \int_0^{\pi} \frac{1}{4}(\cos^2\theta+1)\,d\theta$

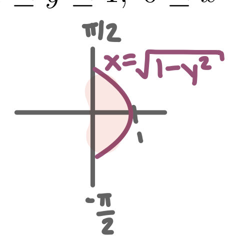
$= \int_0^{\pi} \frac{1}{4}\left(\frac{1}{2}\cos(2\theta)+\frac{1}{2}+1\right)d\theta$

$= \int_0^{\pi} \frac{1}{8}(\cos(2\theta)+3)\,d\theta$

$= \frac{1}{8}\left[\frac{1}{2}\sin(2\theta)+3\theta\right]_0^{\pi}$

$= \frac{3\pi}{8}$

4. $-1 \le y \le 1;\ 0 \le x \le \sqrt{1-y^2}$



$0 \le r \le 1$

$-\frac{\pi}{2} \le r \le \frac{\pi}{2}$

$\int_{-\pi/4}^{\pi/4}\int_0^1 (r^2\cos^2\theta + r^2)\cdot r\,dr\,d\theta$

$= \int_{-\pi/4}^{\pi/4}\int_0^1 r^3(\cos^2\theta+1)\,dr\,d\theta$

$= \int_{-\pi/4}^{\pi/4} \frac{1}{4}r^4\Big|_0^1(\cos^2\theta+1)\,d\theta$

$= \int_{-\pi/4}^{\pi/4} \frac{1}{4}(\cos^2\theta+1)\,d\theta$

$= \int_{-\pi/4}^{\pi/4} \frac{1}{4}\left(\frac{1}{2}\cos(2\theta)+\frac{1}{2}+1\right)d\theta$

$= \int_{-\pi/4}^{\pi/4} \frac{1}{8}(\cos(2\theta)+3)\,d\theta$

$= \frac{1}{8}\left[\frac{1}{2}\sin(2\theta)+3\theta\right]_{-\pi/4}^{\pi/4}$

$= \frac{3\pi+2}{16}$

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