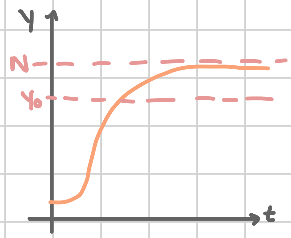
Population Modeling

A population y of a single species with unrestricted growth is given by the differential equation $\frac{dy}{dt} = ky$ for $k > 0$. The issue is that this model allows growth with unlimited resources. There are no caps or restrictions.

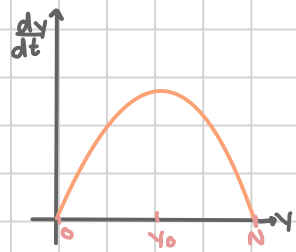
The Logistic Model

In reality there are restrictions like a max population.

unlimited population



max population N



$\frac{dy}{dt}$ with restrictions

$$\frac{dy}{dt} = C\,y(N-y) = \underbrace{CN}_{k}\,y\left(1-\frac{y}{N}\right)$$; N= carrying capacity , k= intrinsic growth rate

Examples:

1. An outbreak of zombies was discovered in the rural City of Sweet Water. The all embracing Mayor Dumass of the city believes that humans and zombies could coexist and wanted laws set up to stop (the still human) people in the city from an all out hunting spree. Renown scientist Dr Madd who survived a recent zombie attack has called for a City Council meeting to convince Mayor Dumass and all administrators of the serious dangers of an unchecked zombie population. Recent estimates give the number of zombies at one thousand and the total city population (both zombies and humans) at forty thousand. If zombie infection occurs at a rate of 0.1 thousand per day per one thousand human per one thousand zombie, how would scientitst Dr Madd convince the City Council that they need to act fast and hit hard on the zombie population before it is too late?

Let H = number of non-infected humans (in thousands)
z = number of infected humans (in thousands)

Then $\frac{dz}{dt}$ is proportional to Hz by the rate k. (Law of Mass Action)
i.e. $\frac{dz}{dt} = k \cdot H \cdot z$ where $k = 0.1$

Since the number of infected has a max population (40 thousand)
$H + z = N$
$H + z = 40$
$H = 40 - z$

Thus $\frac{dz}{dt} = 0.1(40-z)z$ with intial point $z(0) = 1$

general solution:

$\frac{dz}{dt} = 0.1z(40-z)$

$\frac{1}{z(40-z)}dz = 0.1\,dt$

$\int \frac{1}{z(40-z)}dz = \int 0.1\,dt$
① ②

partial fractions

$\int \frac{1/40}{z} + \frac{1/40}{40-z}dz = \int 0.1\,dt$

$\frac{1}{40}\ln|z| - \frac{1}{40}\ln|40-z| = 0.1t + c$

derivative check:
$\frac{1}{40}\cdot\frac{1}{z}\cdot 1 - \frac{1}{40}\cdot\frac{1}{40-z}\cdot -1 = 0.1$ ✓

① $\frac{1}{z(40-z)} = \frac{A}{z} + \frac{B}{40-z}$

$\frac{1}{z(40-z)} = \frac{A}{z}\frac{40-z}{40-z} + \frac{B}{40-z}\cdot\frac{z}{z}$

$1 = A(40-z) + B(z)$
$1 = 40A - Az + Bz$

$z: 0z = Bz - Az$
$c: 1 = 40A$

$A = \frac{1}{40}$
$B = A = \frac{1}{40}$

specific solution:

$z(0) = 1 \quad \Rightarrow (t, z) = (0, 1)$

$\frac{1}{40}\ln|1| - \frac{1}{40}\ln|40-1| = 0.1(0) + c$

$\frac{1}{40}(0) - \frac{1}{40}\ln|39| = 0 + c$

$\frac{-1}{40}\ln(39) = c$

$\frac{1}{40}\ln|z| - \frac{1}{40}\ln|40-z| = 0.1t - \frac{1}{40}\ln(39)$

solve for z:

$\frac{1}{40}\ln|z| - \frac{1}{40}\ln|40-z| = 0.1t - \frac{1}{40}\ln(39)$

$\frac{1}{40}(\ln|z| - \ln|40-z|) = 0.1t - \frac{1}{40}\ln(39)$ factor out $\frac{1}{40}$

$\ln|z| - \ln|40-z| = 4t - \ln(39)$

$\ln\left|\frac{z}{40-z}\right| = 4t - \ln(39)$ $\ln|x| - \ln|y| = \ln\left|\frac{x}{y}\right|$

$$e^{\ln\left|\frac{z}{40-z}\right|} = e^{4t-\ln(39)}$$

$$\frac{z}{40-z} = e^{4t-\ln(39)}$$ $e^{\ln(x)} = x$

$$\frac{z}{40-z} = e^{4t} \cdot e^{-\ln(39)}$$ $e^{a+b} = e^a \cdot e^b$

$$\frac{z}{40-z} = e^{4t} \cdot e^{\ln(\frac{1}{39})}$$ $a\ln(x) = \ln(x^a)$

$$\frac{z}{40-z} = e^{4t} \cdot \frac{1}{39}$$ $e^{\ln(x)} = x$

$$z = \frac{1}{39}e^{4t}(40-z)$$

$$z = \frac{40}{39}e^{4t} - \frac{1}{39}e^{4t}z$$

$$z + \frac{1}{39}e^{4t}z = \frac{40}{39}e^{4t}$$

$$z\left(1+\frac{1}{39}e^{4t}\right) = \frac{40}{39}e^{4t}$$

$$z = \frac{40e^{4t}}{39} \cdot \frac{1}{\left(1+\frac{1}{39}e^{4t}\right)}$$

$$z = \frac{40e^{4t}}{39+e^{4t}}$$

When does the infected population over take the non-infected? i.e. $z = 20$

the expression above are equalities so pick the easiest to plug $z = 20$ into

$$\frac{z}{40-z} = e^{4t} \cdot \frac{1}{39}$$

$$\frac{20}{40-20} = \frac{1}{39}e^{4t}$$

$$1 = \frac{1}{39}e^{4t}$$

$$39 = e^{4t}$$

$$\ln(39) = 4t$$

$t = \frac{1}{4}\ln(39) \approx 0.91589$ days

2. A population of fish grows with growth constant (intrinsic growth rate) of 0.5 in a lake of carrying capacity of 10 thousand is modeled by the logistic differential equation $\frac{dp}{dt} = 0.5p(1-\frac{p}{10})$ where p is its population measured in thousands. If the initial population is 5 thousand, find a formula for $p(t)$.

$$\underbrace{\frac{dp}{dt} = 0.5p(1-\frac{p}{10})}$$

separable diff. eq.

$$\frac{1}{p(1-\frac{p}{10})} \cdot dp = 0.5dt$$

$$\int \frac{1}{p(1-\frac{p}{10})} dp = \int 0.5dt$$

① ②

partial fractions

$$\int \frac{1}{p} + \frac{1}{10-p} dp = \int 0.5 dt$$

general solution:

$$\ln|p| - \ln|10-p| = 0.5t + c$$

specific solution when $p(0) = 5$:

$$\ln|5| - \ln|10-5| = 0.5(0) + c$$

$$\ln|5| - \ln|5| = c$$

$$0 = c$$

solve for p:

$$\ln|p| - \ln|10-p| = 0.5t$$

$$\ln\left|\frac{p}{10-p}\right| = 0.5t$$

$\ln|x| - \ln|y| = \ln\left|\frac{x}{y}\right|$

$$\frac{p}{10-p} = e^{0.5t}$$

$e^{\ln(x)} = x$

$$p = e^{0.5t}(10-p)$$

$$p = 10e^{0.5t} - pe^{0.5t}$$

$$p + pe^{0.5t} = 10e^{0.5t}$$

$$p(1+e^{0.5t}) = 10e^{0.5t}$$

$$p = \frac{10e^{0.5t}}{1+e^{0.5t}}$$

① $\frac{10}{p(10-p)} = \frac{A}{p} + \frac{B}{10-p}$

$$10 = A(10-p) + Bp$$

p: $B - A = 0$

c: $10A = 10$

$A = 1$, $B = 1$

notice: this problem used 3 topics

- separable differential equations
- partial fraction decomposition
- logarithmic / exponential rules

Exit Ticket Separable Differential Equations

Separable Differential Equations The solution to the separable differential equations $\frac{dy}{dx} = p(x)q(y)$ is

$$\frac{dy}{dx} = p(x)q(y)$$
$$\frac{1}{q(y)}dy = p(x)dx$$
$$\int \frac{1}{q(y)}dy = \int p(x)dx$$

Find the general solution to the following differential equations:

1. $\frac{dy}{dx} = 6y^2x$

$$\int \frac{1}{y^2}dy = \int 6x\,dx$$
$$-y^{-1} = 3x^2 + c$$
$$\frac{1}{y} = -3x^2 - C$$
$$\frac{1}{-3x^2 - C} = y$$

2. $y' = \frac{3x^2 + 4x - 4}{2y - 4}$

$$\frac{dy}{dx} = \frac{1}{2y-4}(3x^2+4x-4)$$
$$\int 2y-4\,dy = \int 3x^2+4x-4\,dx$$
$$y^2 - 4y = x^3 + 2x^2 - 4x + C$$

3. $y' = 2xe^{-y} - 4e^{-y}$

$$\frac{dy}{dx} = e^{-y}(2x-4)$$
$$\int e^y dy = \int 2x-4\,dx$$
$$e^y = x^2 - 4x + c$$
$$y = \ln(x^2 - 4x + c)$$

4. $\frac{dy}{dt} = \frac{\cos^2(y)}{y}$

$$\int y \cdot \frac{1}{\cos^2(y)}dy = \int 1\,dt$$
$$\int y \cdot \sec^2(y)\,dy = \int 1\,dt$$
$u = y \quad dv = \sec^2(y)dy$

$du = dy \quad v = \tan(y)$

$$y\tan(y) - \int \tan(y)dy = t + c$$
$$y\tan(y) - \int \frac{\sin(y)}{\cos(y)}dy = t + c$$
$u = \cos(y)$

$du = -\sin(y)dy$

$$y\tan(y) + \int \frac{1}{u}du = t + c$$
$$y\tan(y) + \ln|\cos(y)| = t + c$$