First Order Linear Differential Equations

Q: Solve the initial value problem: $xy' + y = e^{2x}$ and $y(1) = 0$.

Does the left hand side remind you of anything? A derivative rule?
product rule: $(fg)' = f'g + g'f$

general solution:

$xy' + y = e^{2x}$

$\frac{d}{dx}(xy) = e^{2x}$

$\int (xy)' \, dx = \int e^{2x} \, dx$

$\int f' \, dx = f + c$

$xy = \frac{1}{2} e^{2x} + c$

$y = \frac{1}{2x} e^{2x} + \frac{c}{x}$

specific solution:

@ $y(1) = 0$

$0 = \frac{1}{2(1)} e^{2(1)} + \frac{c}{(1)}$

$0 = \frac{1}{2} e^2 + c$

$c = -\frac{1}{2} e^2$

simplify:

$y = \frac{1}{2x} e^{2x} + \frac{(-e^2/2)}{x}$

$y = \frac{1}{2x} e^{2x} - \frac{e^2}{2x}$

$y = \frac{1}{2x}(e^{2x} - e^2)$

First Order Linear Differential Equations

Standard form: $y' + A(x)y = B(x)$

The Key to solve it is to have something like $\frac{d}{dx}(\alpha(x)y)$ on the left hand side (LHS) and a function only in terms of x on the right hand side (RHS).

$$\frac{d}{dx}(\alpha(x)y) = \alpha(x)y' + \alpha'(x)y$$

In order to get $y' + A(x)y$ to resemble $\frac{d}{dx}(\alpha(x)y)$ we must multiply by $\alpha(x)$ on both sides:

$$\alpha(x)y' + \alpha(x)A(x)y = \alpha(x)B(x)$$

For the two to be equivalent we need a function $\alpha(x)$ such that:

$$\alpha'(x) = \alpha(x)A(x)$$

We can use separation of variables to solve this:

$$\frac{d\alpha}{dx} = \alpha A(x)$$

$$\frac{1}{\alpha} d\alpha = A(x)\,dx$$

$$\int \frac{1}{\alpha} d\alpha = \int A(x)\,dx$$

$$\ln|\alpha| = \int A(x)\,dx$$

$$|\alpha| = e^{\int A(x)dx}$$

All this to say:

$$\alpha = e^{\int A(x)dx}$$

We can now integrate both sides:

$$\alpha(x)y' + \alpha(x)A(x)y = \alpha(x)B(x)$$

$$\frac{d}{dx}(\alpha(x)y(x)) = \alpha(x)B(x)$$

$$\int \frac{d}{d(x)}(\alpha(x)y(x))\,dx = \int \alpha(x)B(x)\,dx$$

$$\alpha(x)y(x) = \int \alpha(x)B(x)\,dx$$

$$y(x) = \frac{1}{\alpha(x)} \int \alpha(x)B(x)\,dx$$

All this to say:
$y(x) = \frac{1}{\alpha(x)}\left[\int \alpha(x)B(x)\,dx + c\right]$
is the general solution.

Steps to solve:

I. Write the equation in standard form $y' + A(x)y = B(x)$

II. Compute integrating factor $\alpha(x) = e^{\int A(x)dx}$

III. Find the general solution $y(x) = \frac{1}{\alpha(x)}\left[\int \alpha(x)B(x)dx + c\right]$

IV. If required, find a particular solution.

Examples:

1. $y' - (\tan(x))y = 1$; $y(0) = 3$

   I. $y' + (-\tan(x)) \cdot y = 1$
   $A(x) = -\tan(x)$; $B(x) = 1$

2. $x\frac{dy}{dx} + 5y = \frac{e^x}{x^3}$

   I. $y' + \frac{5}{x} \cdot y = \frac{e^x}{x^4}$
   $A(x) = \frac{5}{x}$; $B(x) = \frac{e^x}{x^4}$

II. $\alpha(x) = e^{\int -\tan(x)dx}$

$= e^{\ln|\cos(x)|}$

$= \cos(x)$

III. $y(x) = \frac{1}{\cos(x)}\left[\int (\cos(x))\cdot(1)\,dx + c\right]$

$= \frac{1}{\cos(x)}\left[\sin(x) + c\right]$

$= \frac{\sin(x)}{\cos(x)} + \frac{c}{\cos(x)}$

$= \tan(x) + \frac{c}{\cos(x)}$

IV. @ $y(0) = 3$

$3 = \tan(0) + \frac{c}{\cos(0)}$

$3 = 1 + \frac{c}{1}$

$3 = c$

$y(x) = \tan(x) + \frac{3}{\cos(x)}$

II. $\alpha(x) = e^{\int \frac{5}{x}dx}$

$= e^{5\ln|x|}$

$= e^{\ln|x^5|}$

$= x^5$

III. $y(x) = \frac{1}{x^5}\left[\int x^5 \cdot \frac{e^x}{x^4}\,dx + c\right]$

$= \frac{1}{x^5}\left[\int xe^x\,dx + c\right]$

$u = x \qquad dv = e^x dx$

$du = 1dx \qquad v = e^x$

$= \frac{1}{x^5}\left[xe^x - \int e^x dx + c\right]$

$= \frac{1}{x^5}\left[xe^x - e^x + c\right]$

$= \frac{e^x}{x^4} - \frac{e^x}{x^5} + \frac{c}{x^5}$

$y(x) = \frac{e^x}{x^4} - \frac{e^x}{x^5} + \frac{c}{x^5}$