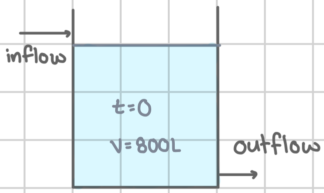
Mixing Tank Problem

Examples:

1. (a) A tank contains 800L of fresh water. Brine that contains 0.05kg of salt per liter enters the tank at a rate of 40L/min. Brine is drained from the tank at the same rate of 40L/min. Find an expression for the amount of salt in the tank at any time t.
   (b) What if the brine is drained out at a slower rate of 10L/min.? Find an expression for the amount of salt in the tank at any time t in this case.



$$\frac{dy}{dt} = \text{salt in} - \text{salt out}$$
$$= (\text{concentration in})(\text{rate in}) - (\text{concentration out})(\text{rate out})$$
$$= (\text{con. in})(\text{rate in}) - \frac{\text{amount of salt in tank}, y(t)}{\text{volume of solution}, v(t)} \cdot (\text{rate out})$$

(a) Outflow rate = 40L/min.

$$v(t) = 800L + \underbrace{40t\ L/min}_{\text{rate in}} - \underbrace{40t\ L/min}_{\text{rate out}}$$

$$\frac{dy}{dt} = (0.05)(40) - \frac{y}{800}(40)$$
$$= 2 - \frac{y}{20}$$
$$\frac{dy}{dt} = \frac{40-y}{20}$$

$$\frac{1}{40-y}\,dy = \frac{1}{20}\,dt$$
$$\int \frac{1}{40-y}\,dy = \int \frac{1}{20}\,dt$$
$$-\ln|40-y| = \frac{1}{20}t + c$$
$$\ln|40-y| = -\frac{1}{20}t - c$$
$$|40-y| = e^{-\frac{1}{20}t - c}$$
$$40-y = ke^{-\frac{1}{20}t}$$
$$y = 40 - ke^{-\frac{1}{20}t}$$

@ $y(0)=0$

$$0 = 40 - ke^{-\frac{1}{20}(0)}$$
$$0 = 40 - k(1)$$
$$k = 40$$

$$y = 40 - 40e^{-\frac{1}{20}t}$$

(b) Outflow rate = 10L/min.

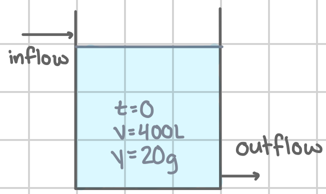
$$v(t) = 800 + 40t - 10t$$

$$\frac{dy}{dt} = (0.05)(40) - \frac{y}{800+30t}(10)$$
$$= 2 - \frac{1}{80+3t}y$$
$$\frac{dy}{dt} + \frac{1}{80+3t}y = 2$$ ← linear first order

1. $A(x) = \frac{1}{80+3t}$; $B(t) = 2$
2. $\alpha(t) = e^{\int \frac{1}{80+3t}dt}$
$$= e^{\frac{1}{3}\ln|80+3t|}$$
$$= e^{\ln(80+3t)^{1/3}}$$
$$= (80+3t)^{1/3}$$
3. $y(t) = \frac{1}{\alpha(t)}\left[\int \alpha(t)B(t)\,dt + c\right]$
$$= \frac{1}{(80+3t)^{1/3}}\left[\int (80+3t)^{1/3}(2)\,dt + c\right]$$
$$= \frac{1}{(80+3t)^{1/3}}\left[\frac{(80+3t)^{4/3}}{2} + c\right]$$
$$= \frac{1}{2}(80+3t) + (80+3t)^{-1/3}c$$
4. @ $y(0) = 0$
$$0 = \frac{1}{(80+3\cdot 0)^{1/3}}\left[\frac{(80+3\cdot 0)^{4/3}}{2} + c\right]$$
$$0 = \frac{1}{2}(80)^{4/3} + c$$
$$c = -\frac{1}{2}(80)^{4/3}$$

$$y = \frac{1}{2}(80+3t) - \frac{1}{2}(80)^{4/3}(80+3t)^{-1/3}$$

2. A tank with a capacity of 400L is full of a mixture of water and chlorine with a concentration of 0.05 g/L. Chlorinated water with a concentration of 0.01 g/L is pumped into the tank at a rate of 4 L/s. The mixture is kept stirred and is pumped out at a rate of 5 L/s. Find the amount of chlorine in the tank as a function of time.



given: concentration in = 0.01g/L
rate in = 4 L/s
rate out = 5 L/s
volume = 400 + 4t - 5t
concentration out = $\frac{y(t)}{400-t}$

$$\frac{dy}{dt} = (\text{con. in})(\text{rate in}) - (\text{con. out})(\text{rate out})$$
$$= (0.01)(4) - \left(\frac{y}{400-t}\right)(5)$$

$$\frac{dy}{dt} + \frac{5}{400-t} \cdot y = 0.04$$

1. $A(x) = \frac{5}{400-t}$; $B(t) = 0.04$

2. $$\alpha(t) = e^{\int \frac{5}{400-t} dt}$$
$$= e^{-5\ln|400-t|}$$
$$= e^{\ln(400-t)^{-5}}$$
$$= (400-t)^{-5}$$

3. $$y(t) = \frac{1}{\alpha(t)}\left[\int \alpha(t) B(t)\, dt + c\right]$$
$$= \frac{1}{(400-t)^{-5}}\left[\int (400-t)^{-5}(0.04)\,dt + c\right]$$
$$= (400-t)^5\left[\frac{0.04}{-4}(400-t)^{-4} + c\right]$$
$$= -0.1(400-t)^5 + (400-t)^5 c$$

4. @ $y(0) = 0$
$$0 = (400-0)^5\left[-0.01(400-0)^{-4} + c\right]$$
$$0 = -0.01(400)^{-4} + c$$
$$c = 0.01(400)^{-4}$$

$$y = -0.1(400-t)^5 + (400-t)^5\, 0.01(400)^{-4}$$