10360 Exam 02 Review Name ______________________________

1. Find y in terms of x: $\dfrac{dy}{dx} = \dfrac{e^{4y}}{x^3}$; $y(1) = 0$.

$$\frac{1}{e^{4y}}dy = \frac{1}{x^3}dx$$

$$\int e^{-4y}dy = x^{-3}dx$$

$$\frac{1}{-4}e^{-4y} = \frac{1}{-2}x^{-2} + c$$

check $\longrightarrow$ $\frac{1}{\cancel{-4}}e^{-4y}\cdot\cancel{-4} = \frac{1}{\cancel{-2}}x^{-3}\cdot\cancel{(-2)} + 0$

@ $y(1)=0$ $(1,0)$ $\quad \frac{1}{-4}e^{-4(0)} = -\frac{1}{2}(1)^{-2} + c$

$$-\frac{1}{4}(1) = -\frac{1}{2} + c$$

$$\frac{1}{4} = c$$

$$-4\cdot\ -\frac{1}{4}e^{-4y} = -\frac{1}{2}x^{-2} + \frac{1}{4}$$

$$\left(-\frac{1}{4}e^{-4y} = -\frac{1}{2}x^{-2} + \frac{1}{4}\right)\cdot -4$$

$$\ln(e^{-4y}) = \ln(2x^{-2} - 1)$$

$$-4y = \ln(2x^{-2} - 1)$$

$$y = -\frac{1}{4}\ln(2x^{-2} - 1)$$

2. Water is flowing in and out of a tank according to the rate $\sin^3(2t)$ m^3/hr. If the tank is filled with 2 m^3 of water, find the amount of water at time t.

initially

$$V(t) = \int V'(t)\,dt$$

$$= \int \sin^3(2t)\,dt$$

$$= \int \sin(2t)\cdot\sin^2(2t)\,dt$$

$$= \int \sin(2t)(1-\cos^2(2t))\,dt$$

$u = \cos(2t)$

$du = -\sin(2t)\cdot 2\,dt$

$-\frac{1}{2}du = \sin(2t)\,dt$

$$= -\frac{1}{2}\int 1 - u^2\,du$$

$$= -\frac{1}{2}\left(u - \frac{1}{3}u^3\right) + c$$

$$= -\frac{1}{2}\cos(2t) + \frac{1}{6}\cos^3(2t) + c$$

@ $(0,2)$ $\quad V(0) = -\frac{1}{2}\cos(2\cdot 0) + \frac{1}{6}\cos^3(2\cdot 0) + c$

$$2 = -\frac{1}{2}(1) + \frac{1}{6}(1) + c$$

$$\frac{12}{6} + \frac{3}{6} - \frac{1}{6} = c = \frac{14}{6}\quad \frac{7}{3}$$

$$V(t) = -\frac{1}{2}\cos(2t) + \frac{1}{6}\cos^3(2t) + \frac{7}{3}$$

3. Find the equation of $y = f(t)$ if its slope is given by $\sin(3t)\sin(4t)$ and that it passes through the point $(\pi, 5)$.

$$f(t) = \int \sin(3t)\sin(4t)\,dt$$

$$\sin A \sin B = -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$$

$$= \int -\frac{1}{2}\left[\cos(7t) - \cos(-t)\right] dt$$

$$= -\frac{1}{2}\left[\frac{1}{7}\sin(7t) + \sin(-t)\right] + c$$

$$f(\pi) = -\frac{1}{2}\left[\frac{1}{7}\sin(7\pi) + \sin(-\pi)\right] + c$$

$$\sin(7\pi) = \sin(\pi) = 0$$

$$= \sin(-\pi) = 0$$

$$5 = -\frac{1}{2}\left[\frac{1}{7}\cdot 0 + 0\right] + c$$

$$5 = C$$

$$f(t) = -\frac{1}{14}\sin(7t) - \frac{1}{2}\sin(-t) + 5$$

4. Evaluate $\int_0^{\pi/4} 4\sin^4(4x)\,dx$.

$$= \int_0^{\pi/4} 4\left(\sin^2(4x)\right)^2 dx$$

$$\sin^2(\theta) = \tfrac{1}{2}\left[1 - \cos(2\theta)\right]$$

$$= \int_0^{\pi/4} 4\left[\tfrac{1}{2}\left(1 - \cos(8x)\right)\right]^2 dx$$

$$= \int_0^{\pi/4} 4\left[\tfrac{1}{4}\left(1 - \cos(8x)\right)^2\right] dx$$

$$= \int_0^{\pi/4} \left(1 - \cos(8x)\right)^2 dx$$

$$= \int_0^{\pi/4} 1 - 2\cos(8x) + \cos^2(8x)\, dx$$

$$\cos^2(\theta) = \tfrac{1}{2}\left[1 + \cos(2\theta)\right]$$

$$= \int_0^{\pi/4} 1 - 2\cos(8x) + \tfrac{1}{2}\left(1 + \cos(16x)\right) dx$$

$$= \int_0^{\pi/4} 1 - 2\cos(8x) + \tfrac{1}{2}\cos(16x) + \tfrac{1}{2}\, dx$$

$$= \int_0^{\pi/4} \tfrac{3}{2} - 2\cos(8x) + \tfrac{1}{2}\cos(16x)\, dx$$

$$= \tfrac{3}{2}x - 2\sin(8x)\cdot\tfrac{1}{8} + \tfrac{1}{2}\sin(16x)\cdot\tfrac{1}{16} \Big|_0^{\pi/4}$$

$$= \tfrac{3}{2}x - \tfrac{1}{4}\sin(8x) + \tfrac{1}{32}\sin(16x) \Big|_0^{\pi/4}$$

$$= \tfrac{3}{2}\left(\tfrac{\pi}{4}\right) - \tfrac{1}{4}\underbrace{\sin(2\pi)}_{0} + \tfrac{1}{32}\underbrace{\sin(4\pi)}_{0} - \left(\underbrace{\tfrac{3}{2}(0)}_{0} - \tfrac{1}{4}\underbrace{\sin(0)}_{0} + \tfrac{1}{32}\underbrace{\sin(0)}_{0}\right)$$

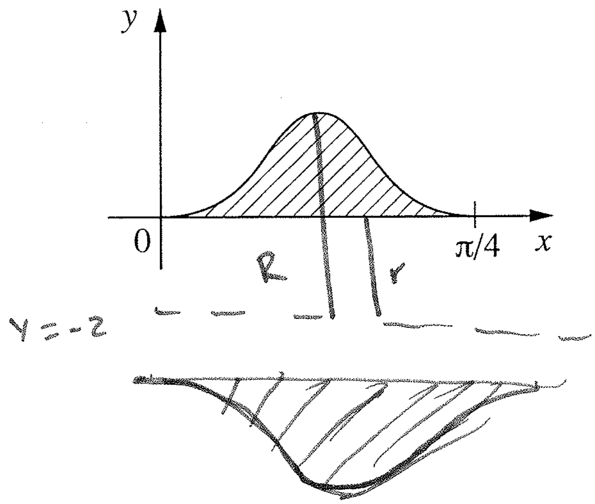
$$= \frac{3\pi}{8}$$

Check derivative:

$$\tfrac{3}{2} - \tfrac{1}{4}\cos(8x)\cdot 8 + \tfrac{1}{32}\cos(16x)\cdot 16$$

$$\tfrac{3}{2} - 2\cos(8x) + \tfrac{1}{2}\cos(16x)\ \checkmark$$

5a. Find the volume of the solid generated when the region below the curve $y = 2\sin^2(4x)$ and $0 \le x \le \frac{\pi}{4}$ is rotated about $y = -2$.



washer: $V = \int_a^b \pi(R^2 - r^2)\,dx$

$= \int_0^{\pi/4} \pi\left((2\sin^2(4x)+2)^2 - (2)^2\right)dx$

$= \pi\int_0^{\pi/4} 4\sin^4(4x) + 8\sin^2(4x) + \cancel{4} - \cancel{4}\,dx$

$= \pi\int_0^{\pi/4} 4(\sin^2(4x))^2 + 8\sin^2(4x)\,dx$

$= \pi\int_0^{\pi/4} 4\left(\frac{1}{2}(1-\cos(8x))\right)^2 + 8\left(\frac{1}{2}(1-\cos(8x))\right)dx$

$= \pi\int_0^{\pi/4} (1-\cos(8x))^2 + 4(1-\cos(8x))\,dx$

$= \pi\int_0^{\pi/4} 1 - 2\cos(8x) + \cos^2(8x) + 4 - 4\cos(8x)\,dx$

$= \pi\int_0^{\pi/4} 5 - 6\cos(8x) + \cos^2(8x)\,dx$

$= \pi\int_0^{\pi/4} 5 - 6\cos(8x) + \frac{1}{2}(1+\cos(16x))\,dx$

$= \pi\int_0^{\pi/4} 5 - 6\cos(8x) + \frac{1}{2} + \frac{1}{2}\cos(16x)\,dx$

$= \pi\int_0^{\pi/4} \frac{11}{2} - 6\cos(8x) + \frac{1}{2}\cos(16x)\,dx$

$R = 2\sin^2(4x) - (-2)$

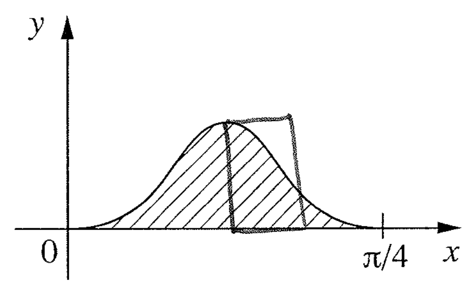
$r = 0 - (-2)$

$2\sin^2(4x) + 2$

$2\sin^2(4x)\ |\ 4\sin^4(4x) + 4\sin^2(4x)$

$+2\ |\ +4\sin^2(4x) + 4$

5b. Consider the solid whose base is the region below the curve $y = 2\sin^2(4x)$ and $0 \le x \le \frac{\pi}{4}$. Find the volume of the solid if the slices of the solid perpendicular to the x-axis are squares.



$V = \int A(x)\,dx$

$= \int_0^{\pi/4} 4\sin^4(4x)\,dx$

$= \int_0^{\pi/4} 4\left(\frac{1}{2}(1-\cos(8x))\right)^2 dx$

$= \int_0^{\pi/4} (1-\cos(8x))^2\,dx$

$= \int_0^{\pi/4} 1 - 2\cos(8x) + \cos^2(8x)\,dx$

$= \int_0^{\pi/4} 1 - 2\cos(8x) + \left(\frac{1}{2}(1+\cos(16x))\right)dx$

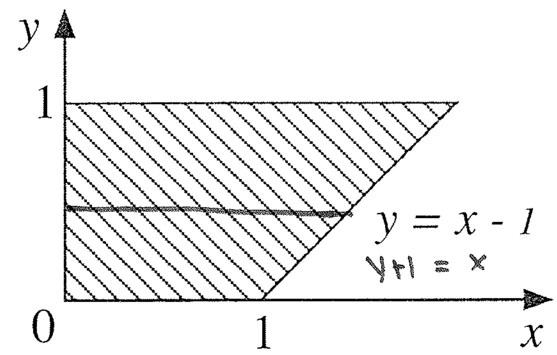
$= \int_0^{\pi/4} 1 - 2\cos(8x) + \frac{1}{2} + \frac{1}{2}\cos(16x)\,dx$

$= \int_0^{\pi/4} \frac{3}{2} - 2\cos(8x) + \frac{1}{2}\cos(16x)\,dx$

$A(x) = (2\sin^2(4x))^2$

$= 4\sin^4(4x)$

6. The shaded region R is given below.



horizontally simply

y-first

Evaluate the following integral:

$R = \{(x,y) \mid 0 \le y \le 1$

$0 \le x \le y+1$

$$\iint_R e^{x+2y}\, dA$$

$$= \int_0^1 \int_0^{y+1} e^{x+2y}\, d\cancel{x}x\, dy$$

$$= \int_0^1 e^{x+2y} \Big|_0^{y+1} dy$$

$$= \int_0^1 e^{(y+1)+2y} \cancel{dy} - e^{(0)+2y}\, dy$$

$$= \int_0^1 e^{3y+1} \cancel{dy} - e^{2y}$$

$$= \frac{1}{3} e^{3y+1} \cancel{\Big|_0^1} - \frac{1}{2} e^{2y} \Big|_0^1$$

$$= \frac{1}{3} e^{3(1)+1} - \frac{1}{3} e^{3(0)+1} - \left(+\frac{1}{2} e^{2(1)} + \frac{1}{2} e^{2(0)}\right)$$

$$= \frac{1}{3} e^4 - \frac{1}{3} e^1 - \left(+\frac{1}{2} e^2 + \frac{1}{2} e^0\right)$$

$$= \frac{1}{3} e^4 - \frac{1}{3} e + \frac{1}{2} e^2 + \frac{1}{2}$$

$$= \frac{1}{3} e^4 + \frac{1}{2} e^2 - \frac{1}{3} e + \frac{1}{2}$$

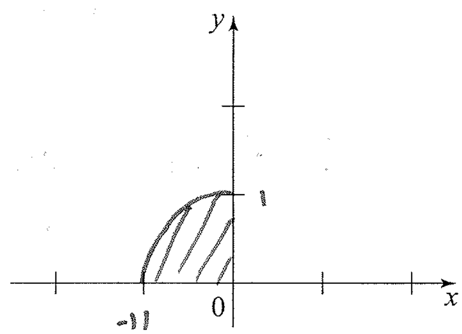
↑ minus

7. The total mass of a metal plate is given by the double integral below

$$\int_{-1}^{0}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^2\, dydx$$

$D = \{(x,y) \mid 0 \le y \le \sqrt{1-x^2}$
$-1 \le x \le 0$

Give a sketch of the shape of the metal plate in the axes below and **shade** the region that the plate covers according to the parametrization in the double integral above. Label the curves in your picture.



Evaluate the integral using any appropriate coordinate system.

$$\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^2\, dydx$$

$R = \{(r,\theta) \mid 0 \le r \le 1$
$\frac{\pi}{2} \le \theta \le \pi\}$

$= \int_{\pi/2}^{\pi}\int_{0}^{1} (r^2)^2 \cdot r\,dr\,d\theta$

$= \int_{\pi/2}^{\pi}\int_{0}^{1} r^5\, dr\, d\theta$

$= \int_{\pi/2}^{\pi} \frac{1}{6} r^6 \Big|_0^1 \, d\theta$

$= \int_{\pi/2}^{\pi} \frac{1}{6}\, d\theta$

$= \frac{1}{6}\theta \Big|_{\pi/2}^{\pi}$

$= \frac{1}{6}\pi - \frac{1}{6}\left(\frac{\pi}{2}\right)$

$= \frac{\pi}{6} - \frac{\pi}{12}$

$= \frac{\pi}{12}$

9. Perform $\displaystyle\int \frac{x^3-3x+5}{x^2+x-2}\,dx$ deg(top) > deg(bottom)

$$
\begin{array}{r l}
 & x - 1 \\
x^2+x-2 & \overline{)\,x^3 - 3x + 5} \\
 & -(x^3 + x^2 - 2x) \\
\hline
 & 0x^3 - x^2 - x + 5 \\
 & -(-x^2 - x + 2) \\
\hline
 & 0x^2 + 0x + 3
\end{array}
$$

$$\int x - 1 + \frac{3}{x^2+x-2}\,dx$$

$$\frac{3}{x^2+x-2} = \frac{3}{(x+2)(x-1)} = \frac{A}{(x+2)} + \frac{B}{(x-1)}$$

$3 = A(x-1) + B(x-2)$

when $x = 1$: $3 = A0 + B(-2)$

$3 = -2B$

$-\frac{3}{2} = B$

when $x = 2$: $3 = A(1) + B(0)$

$3 = A$

$$\int x - 1 + \frac{3}{x+2} + \frac{-3/2}{x-1}\,dx$$

$$= \frac{1}{2}x^2 - x + 3\ln|x+2| - \frac{3}{2}\ln|x-1| + C$$

8. Compute the following integrals. Be sure to use limits if the integral is improper.

8a. $\int \frac{\sec^2 x}{1+2\tan x}\,dx$

$u = 1+2\tan(x)$

$du = 2\sec^2(x)\,dx$

$= \int \frac{1}{u} \cdot \frac{1}{2}\,du$

$= \frac{1}{2}\ln|u| + c$

$= \frac{1}{2}\ln|1+2\tan(x)| + c$

8b. $\int_{-1}^{2} \frac{2}{x^{2/3}}\,dx$ $x^{2/3} = 0$

$x = 0$ improper

$= \int_{-1}^{0} 2x^{-2/3}\,dx + \int_{0}^{2} 2x^{-2/3}\,dy$

$= \lim_{A\to 0^-} \int_{-1}^{A} 2x^{-2/3}\,dx + \lim_{B\to 0^+} \int_{B}^{2} 2x^{-2/3}\,dx$

$= \lim_{A\to 0^-} 6x^{1/3}\Big|_{-1}^{A} + \lim_{B\to 0^+} 6x^{1/3}\Big|_{B}^{2}$

$= \lim_{A\to 0^-}\left[6A^{1/3} - 6(-1)^{1/3}\right] + \lim_{B\to 0^+}\left[6(2)^{1/3} - 6B^{1/3}\right]$

$= 6 \quad + \quad 6(2)^{1/3}$

8c. $\int_{-\infty}^{1} \frac{e^x}{1+e^{2x}}\,dx$

$= \lim_{A\to -\infty} \int_{A}^{1} \frac{e^x}{1+e^{2x}}\,dx$

$= \lim_{A\to -\infty} \int_{A}^{1} \frac{e^x}{1+(e^x)^2}\,dx$ $u = e^x$

$= \lim_{A\to -\infty} \int_{e^A}^{e} \frac{1}{1+u^2}\,du$

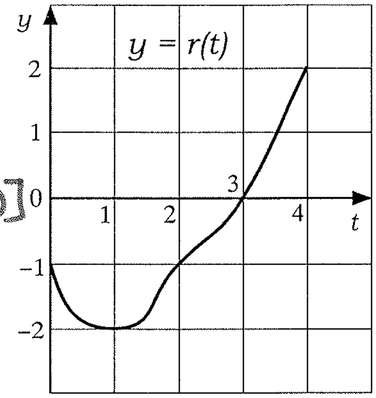
$= \lim_{A\to -\infty} \arctan(u)\Big|_{e^A}^{e}$

$= \lim_{A\to -\infty}\left[\arctan(e) - \arctan(e^a)\right] = \arctan(e) - 0$

10. The graph below shows the rate of change $r(t)$ of the temperature of a room over four hours. Estimate the total change in the temperature of the room over the four hours using the follow numerical methods

(a) Estimate the total change in the temperature of the room over the four hours using **Trapezoidal rule** with 4 equal length segments.

Trapezoidal $\int_a^b f(t)\,dt = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$



$\Delta x = \frac{\cancel{b}-a}{4} = \frac{4-0}{4} = 1$

total change $= \frac{1}{2}\left[r(0) + 2r(1) + 2r(2) + 2r(3) + r(4)\right]$

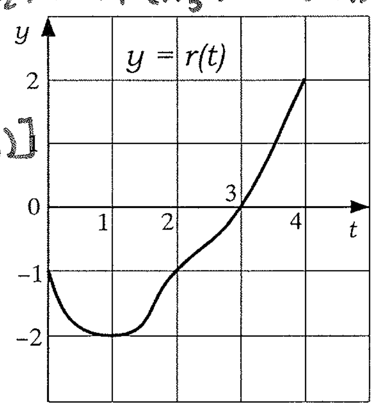
$= \frac{1}{2}\left[-1 + 2(-2) + 2(-1) + 2(0) + 2\right]$

$= \frac{1}{2}\left[-5\right]$

$= -\frac{5}{2}$

(b) Estimate the average rate of change of the temperature in the room over the four hours using **Simpson's rule** with 4 equal length segments.

Simpson's $\int_a^b f(t)\,dt = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$



total change $= \frac{1}{3}\left[r(0) + 4r(1) + 2r(2) + 4r(3) + r(4)\right]$

$= \frac{1}{3}\left[-1 + 4(-2) + 2(-1) + 4(0) + 2\right]$

$= \frac{1}{3}\left[9\right]$

$\cancel{-1-8-3+2}$

$\cancel{-8-2}$

average $\frac{1}{b-a}\int_a^b f(t)\,dt$

$= \frac{1}{4-0}\left[\text{total change}\right]$

$= \frac{1}{4}\left[-\frac{9}{3}\right]$

$= \frac{-3}{4}$

You may find these formulae helpful in the test:

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin A\sin B = -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$$