Numerical Methods

The last few days we have been discussing differential equations and how to solve them. But what happens when we encounter one we can't solve?

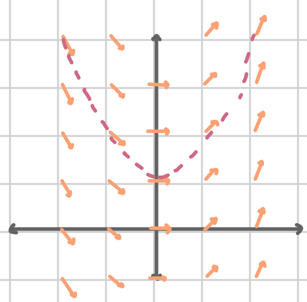
If we are given a differential equation with an initial value then we can estimate the solution. Since $\frac{dy}{dx}$ is a fancy way of saying slope, we can draw a graph with the slope of our function at each point.

Slope Field

Slope field: at each point we have an "arrow" that describes the slope at that point, $z = F(x,y) = \frac{dy}{dx}$.

Graph the slope field for $y' = x$.

x	y	y'
-2		-2
-1		-1
0		0
1		1
2		2



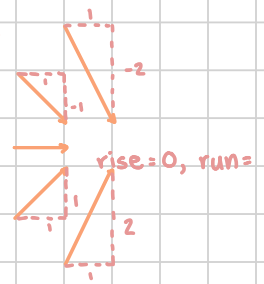
$y'(-2) = -2$: slope of $\frac{-2}{1}$

$y'(-1) = -1$: slope of $\frac{-1}{1}$

$y'(0) = 0$: slope of $\frac{0}{1}$

$y'(1) = 1$: slobe of $\frac{1}{1}$

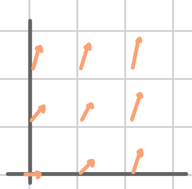
$y'(2) = 2$: slope of $\frac{2}{1}$

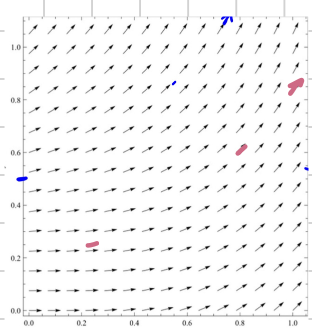


We can use differential equations to create more complex slope fields.

Graph the slope field for $y' = x^2 + y^2$.

y \ x	0	0.5	1
0	0	0.25	1
0.5	0.25	0.5	1.25
1	1	1.25	2





Using a computer generated slope field for $y' = x^2 + y^2$, we can sketch the solution of the initial value problem $y' = x^2 + y^2$, $y(0) = 0.5$

Euler's Method

We use this information to estimate our solution to the differential equation. We start at our initial point, find our slope at that point, graph that slope, and repeat.

Let's start with a differential equation we know how to solve:

Consider the initial value problem $y' = 2x-3$, $y(0)=3$.

Specific Solution:

$\int y'\,dx = \int 2x-3\,dx$
$= x^2 - 3x + c$

@ $y(0)=3$
$3 = 0^2 - 3(0) + c$

$y = x^2 - 3x + 3$

Estimate:

slope at $(0,3)$:
$y' = 2(0)-3 = -3$
$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{-3}{1}$

$x=0.5$: $y' = 2(0.5)-3 = -2$
slope $\frac{-2}{1}$

$x=1$: $y' = 2(1)-3 = -1$
slope $\frac{-1}{1}$

$x=1.5$: $y' = 2(1.5)-3 = 0$
slope $\frac{0}{1}$

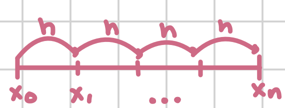
$x=2$: $y' = 2(2)-3 = 1$
slope $\frac{1}{1}$

$x=2.5$: $y' = 2(2.5)-3 = 2$
slope $\frac{2}{1}$

The orange line is the specific solution to the differential equation. The pink line uses the slopes we found above with a "step" size of 0.5. The smaller the step size, the better the estimate.

We can write Euler's method as an algorithm:

__Euler's Method__
Consider the initial-value problem $y' = f(x,y)$, $y(x_0) = y_0$. Using the step size h,
$x_n = x_{n-1} + h$
$y_n = y_{n-1} + h\, f(x_{n-1}, y_{n-1})$



i.e. $h = \frac{x_n - x_0}{n}$

Now let's try a differential equation we don't know how to solve:

Consider the initial value problem $y' = x^2 + y^2$, $y(0) = \frac{1}{2}$. Estimate $y(0.1)$, $y(0.2)$, $y(0.3)$.

Record points:
$(x_0, y_0) = (0, \frac{1}{2})$
$(x_1, y_1) = (0.1, \underline{0.525})$
$(x_2, y_2) = (0.2, \underline{0.5536})$
$(x_3, y_3) = (0.3, \underline{\;?\;})$

secretly given to us by what we are asked to find

$x_1 = x_0 + h$
$= 0 + 0.1 = 0.1$

$y_1 = y_0 + h(x_0^2 + y_0^2)$
$= \frac{1}{2} + (0.1)((0)^2 + (\frac{1}{2})^2)$
$= \frac{1}{2} + (0.1)(\frac{1}{4}) = 0.525$

$x_2 = 0.2$

$y_2 = y_1 + h(x_1^2 + y_1^2)$
$= 0.525 + 0.1((0.1)^2 + (0.525)^2)$
$= 0.5536$

Examples:

1. Consider the initial value problem $y' = x^2 - y^2$, $y(-1) = 2$. Use a step size of 0.1 to estimate $y(-0.8)$.

Record Points:

$(x_0, y_0) = (-1, 2)$
$(x_1, y_1) = (-0.9, 1.7)$
$(x_2, y_2) = (-0.8, 1.492)$
$(x_3, y_3) =$

$$x_1 = x_0 + h = -1 + 0.1 = -0.9$$

$$\begin{aligned} y_1 &= y_0 + h \cdot f(x_0, y_0) \\ &= 2 + (0.1)[(-1)^2 - (2)^2] \\ &= 2 + (0.1)[1 - 4] \\ &= 2 + (0.1)(-3) \\ &= 2 - 0.3 = 1.7 \end{aligned}$$

As you can see, these problems require calculators or only 2-steps.

$$\begin{aligned} x_2 &= x_1 + h \\ &= -0.9 + 0.1 \\ &= -0.8 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + h \cdot f(x_1, y_1) \\ &= 1.7 + (0.1)[(-0.9)^2 + (1.7)^2] \\ &= 1.492 \end{aligned}$$

2. The weight w in kilograms of a kind of tropical fungus is modeled by the differential equation $\frac{dw}{dt} = \frac{\sqrt{w}}{t^2+1}$, $w(1) = 4$. Here t denotes the time measured in weeks. Estimate the weight of the fungus a $t = 2.5$ weeks using Euler's method with three steps.

given: $\frac{dw}{dt} = \frac{\sqrt{w}}{t^2+1}$, $(x_0, y_0) = (1, 4)$

step size: $h = \frac{x_n - x_0}{n} = \frac{2.5 - 1}{3} = \frac{1.5}{3} = \frac{1}{2}$

record points:

$(x_0, y_0) = (1, 4)$
$(x_1, y_1) = (1.5, 4.5)$
$(x_2, y_2) = (2, 4.8264)$
$(x_3, y_3) = (2.5, 5.0461)$

$$\begin{aligned} x_1 &= x_0 + h \\ &= 1 + \tfrac{1}{2} \\ &= 1.5 \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + h \cdot f(x_0, y_0) \\ &= 4 + \tfrac{1}{2}\left[\frac{\sqrt{4}}{(1)^2+1}\right] \\ &= 4 + \tfrac{1}{2}\left[\tfrac{2}{2}\right] = 4.5 \end{aligned}$$

$$\begin{aligned} x_2 &= x_1 + h \\ &= 1.5 + \tfrac{1}{2} \\ &= 2 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + h \cdot f(x_1, y_1) \\ &= 4.5 + \tfrac{1}{2}\left[\frac{\sqrt{4.5}}{(1.5)^2+1}\right] \\ &= 4.8264 \end{aligned}$$

$$\begin{aligned} x_3 &= x_2 + h \\ &= 2 + \tfrac{1}{2} \\ &= 2.5 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + h \cdot f(x_2, y_2) \\ &= 4.8264 + \tfrac{1}{2}\left[\frac{\sqrt{4.8264}}{(2)^2+1}\right] \\ &= 5.0461 \end{aligned}$$