Partial Derivative

Partial Derivatives

Recall that given a function of one variable, $f(x)$, the derivative, $f'(x)$, represents the rate of change of the function as x changes. The issue, we have more than one variable to vary. If we allow more than one to vary then we have an infinite number of ways we can change them: same speed, one faster than the other, different degrees of faster, etc. In this section we concentrate on changing only one variable at a time as the remaining variable(s) are held fixed.

In practice, the partial derivative of $f = f(x,y)$ with respect to x is the derivative of f with respect to x while treating all other variable(s) as a constant. We denote the partial with respect to x as f_x. We can also define a partial with respect to y similarly: take the derivative of f with respect to y while treating all other variables as constants. This definition can be extended to a function with more than two variables. You can also take higher partial derivatives; $(f_x)_x$, $(f_x)_y$, $(f_y)_y$, $(f_y)_x$.

Alternative notation: $f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(f(x,y)) = z_x = \frac{\partial z}{\partial x} = D_x f$.

Examples:

1. Find the partial derivatives f_x, f_y for $f(x,y) = x^2 + y^2 + xy$.

$f_x = 2x + 0 + y$ — x^2 is done as normal, y^2 is a constant, xy has a constant of y

$f_y = 0 + 2y + x$ — x^2 is a constant, y^2 is as normal, xy has a constant of x

You can continue this pattern and take $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, $(f_y)_y$.

2. Find the higher partials of $f(x,y) = x^2 + y^2 + xy$

$(f_x)_x = 2$ — x partial of $f_x = 2x + y$ i.e. y is a constant

$(f_x)_y = 1$ — y partial of $f_x = 2x + y$ i.e. $2x$ is a constant

$(f_y)_x) = 1$ — x partial of $f_y = 2y + x$ i.e. $2y$ is a constant

$(f_y)_y = 2$ — y partial of $f_y = 2y + x$ i.e. x is a constant

Limit Definition

Recall the limit definition of a derivative: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

We have analogous definitions for partial derivatives:

Let $f(x,y)$ be a function of two variables. Then
(i) the partial derivative of f with respect to x is

$$\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

(i) the partial derivative of f with respect to y is

$$\frac{\partial f}{\partial y}(x,y) = f_x(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Sometimes this is referred to as the instaneous rate of change of f in the x (or y)-direction.

Examples:

1. Evaluate the following limits:

(i) $\lim_{h \to 0} \frac{\ln(3x+2(y+h)) - \ln(3x+2y)}{h}$

y has $+h$ $\Rightarrow \frac{\partial}{\partial y}$

$f(x,y) = \ln(3x+2y)$

$f_y = \frac{1}{3x+2y} \cdot 2$

(i) $\lim_{h \to 0} \frac{\ln(3(x+h)+2y) - \ln(3x+2y)}{h}$

x has $+h$ $\Rightarrow \frac{\partial}{\partial x}$

$f(x,y) = \ln(3x+2y)$

$f_x = \frac{1}{3x+2y} \cdot 3$

Exit Ticket Euler's Method

> **Euler's Method** Consider the initial value problem
>
> $$y' = f(x, y);\ y(x_0) = y_0.$$
>
> Using the step size h
>
> $$\begin{aligned} x_n &= x_{n-1} + h \\ y_n &= y_{n-1} + h \cdot f(x_{n-1}, y_{n-1}) \end{aligned}$$

Find $(x_2, y_2$ for each of the following differential equations using the step size $h = \frac{1}{2}$.

1. $\dfrac{dy}{dx} = 1 + 2xy;\ y(0) = 3$

$x_0 = 0 \qquad y_0 = 3$

$x_1 = \frac{1}{2} \qquad y_1 = 3 + (\frac{1}{2})(1 + 2(0)(3))$

$= 3 + (\frac{1}{2})(1)$

$= 3.5$

$x_2 = 1 \qquad y_2 = \frac{7}{2} + (\frac{1}{2})(1 + 2(\frac{1}{2})(\frac{7}{2}))$

$= \frac{7}{2} + (\frac{1}{2})(\frac{9}{2})$

$= \frac{7}{2} + \frac{9}{4} = \frac{23}{4}$

2. $y' = x^2 + y^2;\ y(1) = 2$

$x_0 = 1 \qquad y_0 = 2$

$x_1 = 1.5 \qquad y_1 = 2 + (\frac{1}{2})((1)^2 + (2)^2)$

$= 2 + (\frac{1}{2})(5)$

$= \frac{4}{2} + \frac{5}{2} = \frac{9}{2}$

$x_2 = 2 \qquad y_2 = \frac{9}{2} + (\frac{1}{2})((\frac{3}{2})^2 + (\frac{9}{2})^2)$

$= \frac{9}{2} + (\frac{1}{2})(\frac{9}{2} + \frac{81}{2})$

$= \frac{9}{2} + (\frac{1}{2})(45) = \frac{54}{2}$

$= \frac{27}{2}$

3. $y' = 3x + 3y^2;\ y(0) = 1$

$x_0 = 0 \qquad y_1 = 1$

$x_1 = \frac{1}{2} \qquad y_1 = 1 + (\frac{1}{2})(3(0) + 3(1)^2)$

$= 1 + (\frac{1}{2})(3)$

$= \frac{5}{2}$

$x_2 = 1 \qquad y_2 = \frac{5}{2} + (\frac{1}{2})(3(1) + 3(\frac{5}{2})^2)$

$= \frac{5}{2} + (\frac{1}{2})(3 + 3(\frac{25}{4}))$

$= \frac{5}{2} + (\frac{1}{2})(\frac{12}{4} + \frac{75}{4})$

$= \frac{5}{2} + (\frac{1}{2})(\frac{87}{4})$

$= \frac{20}{8} + \frac{87}{8} = \frac{107}{8}$

4. $\dfrac{dy}{dt} = \dfrac{\sqrt{y}}{t+1};\ y(1) = 1$

$x_0 = 1 \qquad y_0 = 1$

$x_1 = 1.5 \qquad y_1 = 1 + (\frac{1}{2})(\frac{\sqrt{1}}{1+1})$

$= 1 + (\frac{1}{2})(\frac{1}{2})$

$= 1 + \frac{1}{4} = \frac{5}{4}$

$x_2 = 2 \qquad y_2 = \frac{5}{4} + (\frac{1}{2})(\frac{\sqrt{5/4}}{(3/2)+1})$

$= \frac{5}{4} + (\frac{1}{2})(\frac{\sqrt{5}/2}{5/2})$

$= \frac{5}{4} + (\frac{1}{2})(\frac{\sqrt{5}}{2} \cdot \frac{2}{5})$

$= \frac{5}{4} + (\frac{1}{2})(\frac{\sqrt{5}}{5})$

$= \frac{5}{4} + \frac{\sqrt{5}}{10} = \frac{25 + 2\sqrt{5}}{20}$

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