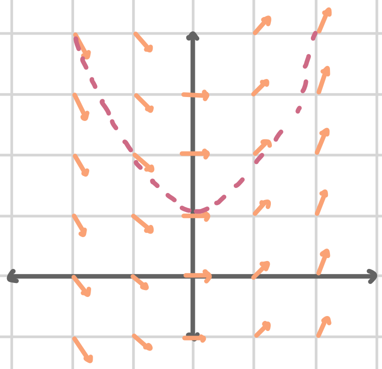
Week 12: April 6^{th}, 2023

Section 9.3: Graphical and Numerical Methods

Slope field: at each point we have an "arrow" that describes the slope at that point, $z = F(x,y) = \frac{dy}{dx}$.

example. Graph the slope field for $y' = x$.

x	y	y'
-2		-2
-1		-1
0		0
1		1
2		2



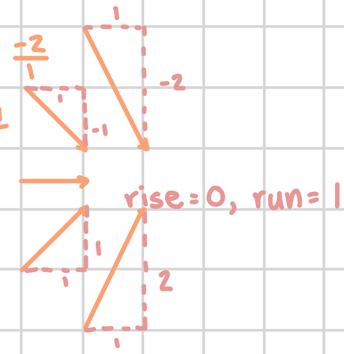
$y'(-2) = -2$: slope of $\frac{-2}{1}$

$y'(-1) = -1$: slope of $\frac{-1}{1}$

$y'(0) = 0$: slope of $\frac{0}{1}$

$y'(1) = 1$: slope of $\frac{1}{1}$

$y'(2) = 2$: slope of $\frac{2}{1}$

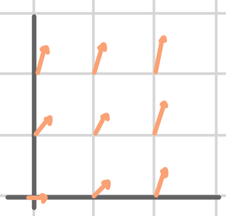


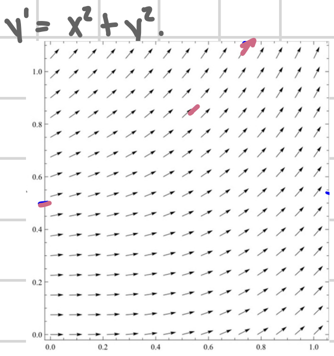
We can use these slopes to sketch $y = \frac{1}{2}x^2 + c$

We can use differential equations to create more complex slope fields.

example. Graph the slope field for $y' = x^2 + y^2$.

y \ x	0	0.5	1
0	0	0.25	1
0.5	0.25	0.5	1.25
1	1	1.25	2





Using a computer generated slope field for $y' = x^2 + y^2$, we can sketch the solution of the initial value problem $y' = x^2 + y^2$, $y(0) = 0.5$

Euler's Method

Consider the initial value problem $y' = 2x - 3$, $y(0) = 3$.

Initial Value:

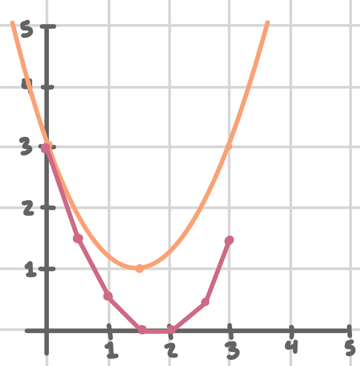
$\int y'\,dx = \int 2x - 3\,dx$

$= x^2 - 3x + C$

@ $y(0) = 3$

$3 = 0^2 - 3(0) + C$

$y = x^2 - 3x + 3$



Approximate:

$(0,3)$: $y' = 2(0) - 3 = -3$
slope $\frac{-3}{1}$

$x = 0.5$: $y' = 2(0.5) - 3 = -2$
slope $\frac{-2}{1}$

$x = 1$: $y' = 2(1) - 3 = -1$
slope $\frac{-1}{1}$

$x = 1.5$: $y' = 2(1.5) - 3 = 0$
slope $\frac{0}{1}$

$x = 2$: $y' = 2(2) - 3 = 1$
slope $\frac{1}{1}$

$x = 2.5$: $y' = 2(2.5) - 3 = 2$
slope $\frac{2}{1}$

The pink graph consists of line segments that approximate the solution to the initial value of $(0,3)$. The slope of the solution at any point is determined by the right-hand side of the differential equation, and the length of the line segment is determined by increasing the x value by 0.5 each time. This is the basis of Euler's Method.

Before we state Euler's Method, let us consider a more complex initial-value problem: $y' = x^2 - y^2$, $y(-1) = 2$.

At the point $(-1,2)$, the slope of the solution is given by $y' = (-1)^2 - 2^2 = -3$, so the slope of the tangent line to the solution at that point is also equal to -3. We take $x_0 = -1$ and $y_0 = 2$. We can use the method of linear approximation y near $(-1,2)$: $L(x) = y_0 + f'(x_0)(x - x_0)$

Using $x_0 = -1$, $y_0 = 2$, and $f'(x_0) = -3$: $L(x) = 2 - 3(x - (-1))$

$= 2 - 3x - 3$

$= -3x - 1$.

Now we choose a step size h and increment x by this value: $x_1 = x_0 + h$.

Pick $h = 0.1$. Then $x_1 = x_0 + h = -1 + 0.1 = -0.9$.

Calculate $y_1 = L(x_1) = -3(-0.9) - 1$

$= 1.7$.

Therefore the approximation for y when $x=-0.9$ is $y=1.7$. We can repeat this algorithm using $x_1=-0.9$ and $y_1=1.7$ to solve for x_2 and y_2:

- the new slope is $y'=(-0.9)^2-(1.7)^2=-2.08$
- $x_2=x_1+h=-0.9+0.1=-0.8$
- $L(x)=y_1+f'(x_1)(x-x_1)$

$$\begin{aligned} &= 1.7-2.08(x-(-0.9)) \\ &= 1.7-2.08x-1.872 \\ &= -2.08x-0.172 \end{aligned}$$

- $y_2=L(x_2)=-2.08(-0.8)-0.172$

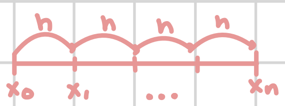
$$= 1.492$$

Euler's Method

Consider the initial-value problem $y'=f(x,y)$, $y(x_0)=y_0$. Using the step size h,

$x_n=x_0+nh$

$y_n=y_{n-1}+h\,f(x_{n-1},y_{n-1})$



Partial Derivatives

def. The partial derivative of $f=f(x,y)$ with respect to x $f_x(x,y)$ and with respect to y $f_y(x,y)$ is the derivative where we pretend all other variables are constants.

example. Find the partial derivatives f_x, f_y for $f(x,y)=x^2+y^2+xy$.

$f_x=2x+0+y$ the first term is as normal, y^2 is just a constant, xy has a constant of y

$f_y=0+2y+x$ x^2 is just a constant, y^2 is your normal derivative, xy has a constant of x

You can continue this pattern and take $(f_x)_x, (f_x)_y, (f_y)_x, (f_y)_y$.

example. Find the higher partials of $f(x,y)=x^2+y^2+xy$

$(f_x)_x=2$, $(f_x)_y=1$, $(f_y)_x=1$, $(f_y)_y=2$