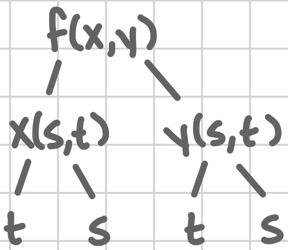
Chain Rule and Implicit Differentation

Chain Rule

The chain rule allows us to take a derivative down a tree of functions:



The derivative down each line is the top over the bottom

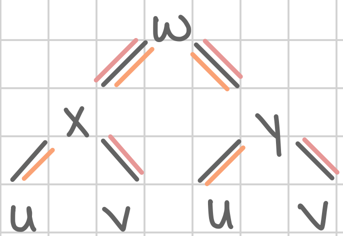
$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$$

This replicates the process of plugging in the parameterizations

Examples:

1. Let $w = 3x\cos(\pi y)$. If $x = u^2 + v^2$, $y = \frac{v}{u}$, find the following partial derivatives at the given point. Use a tree diagram.



$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}$$

when $u=1$ & $v=1$

$$x = u^2 + v^2 = (1)^2 + (1)^2 = 2$$

$$y = \frac{v}{u} = \frac{1}{1} = 1$$

(a) $\frac{\partial w}{\partial u}$ at $u=1$ and $v=1$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial w}{\partial x} = 3\cos(\pi y) = 3\cos(\pi\cdot 1) = -3$$

$$\frac{\partial x}{\partial u} = 2u = 2\cdot 1 = 2$$

$$\frac{\partial w}{\partial y} = -3x\sin(\pi y)\cdot\pi = -3\pi(2)\sin(\pi) = 0$$

$$\frac{\partial y}{\partial u} = -vu^{-2} = -(1)(1)^{-2} = -1$$

$$\frac{\partial w}{\partial u} = (-3)(2) + (0)(-1) = -6 + 0 = -6$$

(b) $\frac{\partial w}{\partial v}$ at $u=1$ and $v=1$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial w}{\partial x} = 3\cos(\pi y) = 3\cos(\pi\cdot 1) = -3$$

$$\frac{\partial x}{\partial v} = 2v = 2\cdot 1 = 2$$

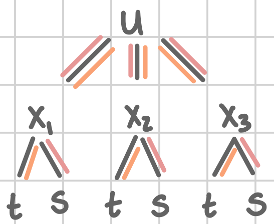
$$\frac{\partial w}{\partial y} = -3\pi x\sin(\pi y) = -3\pi(2)\sin(\pi\cdot 1) = 0$$

$$\frac{\partial y}{\partial v} = \frac{1}{u} = \frac{1}{1} = 1$$

$$\frac{\partial w}{\partial v} = (-3)(2) + (0)(1) = -6 + 0 = -6$$

Make sure you plug x-values in x's, y's in y's, u's in u's, and v's in v's.

2. Let $u=e^{x_1+4x_2-x_3}$. If $x_1=2t-s$, $x_2=t^2$, and $x_3=t+3s$. Find the following partial derivatives at the given points.



when $s=1, t=-1$

$x_1=2(-1)-(1)$
$=-3$

$x_2=(-1)^2$
$=1$

$x_3=-1+3(1)$
$=2$

(i) $\frac{\partial u}{\partial t}$ where $s=1, t=-1$

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t}+\frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t}+\frac{\partial u}{\partial x_3}\frac{\partial x_3}{\partial t}$$

$$=(e^{x_1+4x_2-x_3}\cdot 1)(2)+(e^{x_1+4x_2-x_3}\cdot 4)(2t)+(e^{x_1+4x_2-x_3}\cdot -1)(1)$$

$$=2e^{-3+4(1)-2}+2(-1)e^{-3+4(1)-2}-e^{-3+4(1)-2}$$

$$=2e^{-1}-2e^{-1}-e^{-1}$$

$$=e^{-1}$$

(ii) $\frac{du}{\partial s}$ where $s=1, t=-1$

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial s}+\frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial s}+\frac{\partial u}{\partial x_3}\frac{\partial x_3}{\partial s}$$

$$=(e^{x_1+4x_2-x_3}\cdot 1)(-1)+(e^{x_1+4x_2-x_3}\cdot 4)(0)+(e^{x_1+4x_2-x_3}\cdot -1)(3)$$

$$=-e^{-3+4(1)-2}+0-3e^{-3+4(1)-2}$$

$$=-e^{-1}-3e^{-1}$$

$$=-4e^{-1}$$

3. Given that $ze^{x+2y}+z^2-x-y=0$. Find $z_x=\frac{\partial z}{\partial x}$ and $z_y=\frac{\partial z}{\partial y}$

we assume z is a function of x and y thus we use implicit differentation

$$\frac{\partial}{\partial x}[ze^{x+2y}+z^2-x-y]=\frac{\partial}{\partial x}[0]$$

$$\frac{\partial z}{\partial x}\cdot e^{x+2y}+e^{x+2y}\cdot z+2z\frac{\partial z}{\partial x}-1-0=0$$

$$\frac{\partial z}{\partial x}e^{x+2y}+2z\frac{\partial z}{\partial x}=1-ze^{x+2y}$$

$$\frac{\partial z}{\partial x}(e^{x+2y}+2z)=1-ze^{x+2y}$$

$$\frac{\partial z}{\partial x}=(1-ze^{x+2y})/(e^{x+2y}+2z)$$

$$\frac{\partial}{\partial y}[ze^{x+2y}+z^2-x-y]=\frac{\partial}{\partial y}[0]$$

$$\frac{\partial z}{\partial y}e^{x+2y}+2e^{x+2y}z+2z\frac{\partial z}{\partial y}-0-1=0$$

$$\frac{\partial z}{\partial y}e^{x+2y}+2z\frac{\partial z}{\partial y}=1-2ze^{x+2y}$$

$$\frac{\partial z}{\partial y}(e^{x+2y}+2z)=1-2ze^{x+2y}$$

$$\frac{\partial z}{\partial y}=(1-2ze^{x+2y})/(e^{x+2y}+2z)$$

Exit Ticket Estimating Partial Derivatives

Estimating Partial Derivatives There are three formulas for estimating partial derivatives with respect to x:

- **forward difference:** $\frac{\partial}{\partial x}[f(x,y)] \approx \frac{f(x+h,y)-f(x,y)}{h}$
- **backwards difference:** $\frac{\partial}{\partial x}[f(x,y)] \approx \frac{f(x,y)-f(x-h,y)}{h}$
- **central difference:** $\frac{\partial}{\partial x}[f(x,y)] \approx \frac{f(x+h,y)-f(x-h,y)}{2h}$

Below is a chart that describes the "Feels like" temperature ($F(T,W)$) given the wind speed (W) and air temperature (T), give all estimates of $\frac{\partial F}{\partial W}$ at the given points:

∂w

$W \backslash T$	40	35	30	25
5	36	31	25	19
10	34	27	21	15
15	31	25	19	13
20	30	24	17	11

$h=5$

1. $(35,15)$

(down) forward: $\frac{f(35,20)-f(35,15)}{5} = \frac{24-25}{5} = \frac{-1}{5}$

(up) backward: $\frac{f(35,15)-f(35,10)}{5} = \frac{25-27}{5} = \frac{-2}{5}$

central: $\frac{f(35,20)-f(35,10)}{2h} = \frac{24-27}{10} = \frac{-3}{10}$

2. $(25,20)$

backwards: $\frac{f(25,20)-f(25,15)}{5}$

$= \frac{11-13}{5} = \frac{-2}{5}$

3. $(40,5)$

forward: $\frac{f(40,10)-f(40,5)}{h}$

$= \frac{34-36}{5} = \frac{-2}{5}$

4. $(30,10)$

forward: $\frac{f(30,15)-f(30,10)}{5}$

$= \frac{19-21}{5} = \frac{-2}{5}$

backward: $\frac{f(30,10)-f(30,5)}{5}$

$= \frac{21-25}{5} = \frac{-4}{5}$

central: $\frac{f(30,15)-f(30,5)}{10}$

$= \frac{19-25}{10} = \frac{-6}{10} = \frac{-3}{5}$

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