Linear Approximation

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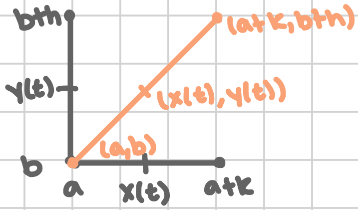
Consider a particle moving from point (a,b) to point $(a+k, b+h)$. If the particle travels at a constant speed and the total duration of the motion is 1 second, find in terms of time (in seconds), a formula for the position (x,y).

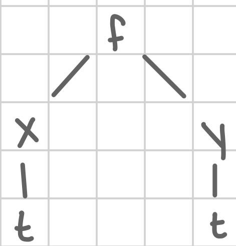


$(x(0), y(0)) = (a,b)$

$(x(1), y(1)) = (a+k, b+h)$

$(x(t), y(t)) = (a+tk, b+ht)$ w/ $\frac{dx}{dt} = k$, $\frac{dy}{dt} = h$

Consider a function $f(x,y)$ such that its first partial derivatives exist for all points near (a,b). If (x,y) is a point on the line segment found above, find a formula for the rate of change of f with respect to t.



$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$= k\frac{\partial f}{\partial x} + h\frac{\partial f}{\partial y}$$

For a small change in time Δt, let the corresponding change in x be from a be Δx, the corresponding change in y from b be Δy and Δf be the corresponding change in f from $f(a,b)$. Then we have $\frac{\Delta f}{\Delta t} \approx \frac{df}{dt}\Big|_{t=0}$. We want to show that $\Delta f \approx \frac{\partial f}{\partial x}(a,b) \cdot \Delta x + \frac{\partial f}{\partial y}(a,b) \cdot \Delta y$ where $\Delta f = f(a+\Delta x, b+\Delta y) - f(a,b)$. This Δf is called the Linear Approximation of change in f when (x,y) changes from (a,b) to $(a+\Delta x, b+\Delta y)$.

$\Delta t \longrightarrow \Delta x \approx \frac{dx}{dt}$, $\Delta y \approx \frac{dy}{dt}$

$\frac{\Delta f}{\Delta t} = \frac{f(t+h) - f(t)}{h}$

$f(t+h) - f(t) \approx \frac{\partial f}{\partial x}(a,b)\,\Delta x + \frac{\partial f}{\partial y}(a,b)\,\Delta y$

Example:

1. A two-variable function $f(x,y)$ has selected values given by

y\x	2.5	3.0	3.5
-1.0	6.0	6.5	8.0
-1.5	6.5	7.0	8.5
-2.0	5.8	6.9	7.8

Using the central difference estimate for $\frac{\partial f}{\partial y}(3.5,-1.5)$ and the estimate of $\frac{\partial f}{\partial x}(3.5,-1.5)$ approximate the value of $f(3.2,-1.1)$.

formula: $f(x,y) \approx f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$

estimates: $\frac{\partial f}{\partial y}(3.5,-1.5) \approx 0.2$, $\frac{\partial f}{\partial x}(3,-1.5) \approx 3$

linear approximation:

$$f(3.2,-1.1) \approx f(3.5,-1.5) + \frac{\partial f}{\partial x}(3.5,-1.5)(3.2-3.5) + \frac{\partial f}{\partial y}(3.5,-1.5)(-1.1+1.5)$$

$$= 8.5 + 3(-0.3) + 0.2(0.4)$$

$$= 7.68$$

2. Let $g(x,y) = \sqrt{4-x^2+y^2}$. Using linear approximation of $g(x,y)$ at $(1,1)$, estimate the following values:

(a) the change in $g(x,y)$ when (x,y) changes from $(1,1)$ to $(1.1,0.8)$

(b) the value of $g(1.1,0.8)$

(c) the percent change in $g(x,y)$ when (x,y) changes from $(1,1)$ to $(1.1,0.8)$

(d) the linearization of $g(x,y)$ at $(1,1)$

(a) $\Delta g = \frac{\partial g}{\partial x}(a,b)\cdot \Delta x + \frac{\partial g}{\partial y}(a,b)\cdot \Delta y$

$\frac{\partial g}{\partial x} = \frac{1}{2}(4-x^2+y^2)^{-1/2}\cdot(-2x)$

$\Big|_{(1,1)} = \frac{1}{2}\frac{1}{\sqrt{4-1^2+1^2}}\cdot(-2\cdot 1)$

$= -\frac{1}{2}$

$\frac{\partial g}{\partial y} = \frac{1}{2}(4-x^2+y^2)^{-1/2}\cdot(2y)$

$= \frac{1}{2}\frac{1}{\sqrt{4-1^2+1^2}}\cdot(2\cdot 1)$

$= \frac{1}{2}$

$\Delta g = -\frac{1}{2}(1.1-1) + \frac{1}{2}(0.8-1)$

$= -\frac{1}{2}(0.1) + \frac{1}{2}(-0.2)$

$= -0.05 - 0.1$

$= -0.15$

(b) $\Delta f = f(a+\Delta x, b+\Delta y) - f(a,b)$

$-0.15 = f(1.1,0.8) - f(1,1)$

$-0.15 = f(1.1,0.8) - \sqrt{4-1^2+1^2}$

$-0.15 = f(1.1,0.8) - 2$

$1.85 = f(1.1,0.8)$

(c) percent change $= 100\cdot\frac{\Delta f}{f(a,b)}$

% change $= 100\cdot\frac{1.85}{2}$

$= 92.5\%$

(d) $g(x,y) \approx g(a,b) + \frac{\partial g}{\partial x}(a,b)(x-a) + \frac{\partial g}{\partial y}(a,b)(y-b)$

$= 2 + (-\frac{1}{2})(x-1) + (\frac{1}{2})(y-1)$

$= 2 - \frac{1}{2}x + \frac{1}{2} + \frac{1}{2}y - \frac{1}{2}$

$= 2 - \frac{1}{2}x + \frac{1}{2}y$

Exit Ticket Chain Rule

Chain Rule Let $z = f(x, y)$, $x = g(s, t)$, and $y = h(s, t)$ be functions of two variables. The partial derivatives $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ can be found by the chain rules:

1. $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$
2. $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$

Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ for the functions below:

1. $z = x^2 + 2xy,\ y = s + t,\ x = s^2 + 4t$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$
$$= (2x+2)(2s) + (2)(1)$$
$$= (2(s^2+4t)+2)(2s)+2)$$
$$= 4s^3 + 16st + 4s + 2$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$
$$= (2x+2)(4) + (2)(1)$$
$$= 8x + 8 + 2$$
$$= 8x + 10$$

2. $z = x\cos(x) + y^2,\ x = 3t + 1,\ y = s^2 + t^2$

3. $z = \frac{x^2 - x}{y^4},\ x = t^3,\ y = cos(2s)$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$
$$= \left[\frac{1}{y^4}(2x-1)\right](0) + \left(-4 \cdot \frac{x^2-x}{y^5}\right)(-2\sin(2s)$$
$$= 0 - 4\left(\frac{t^6-t^3}{\cos^5(2s)}\right)(-2\sin(2s))$$
$$= (4t^6+4t^3) \cdot \frac{\sin(2s)}{\cos^5(2s)}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$
$$= \left[\frac{1}{y^4}(2x-1)\right](3t^2) + \left(-4 \cdot \frac{x^2-x}{y^5}\right)(0)$$
$$= \frac{3t^2}{\cos^4(2s)}(2t^3-1) + 0$$
$$= \frac{6t^5-3t^2}{\cos^4(2s)}$$

4. $z = \sqrt{x^2 + y^2} + \frac{y}{x},\ x = \sin(t),\ y = s^2 + t^2$

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