Sensitivity and Elasticity

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Let $z = f(x,y)$. Set $x=a$, $y=b$. Then the sensitivity of the quantity of z relative to x at (a,b) is measured by $\frac{\partial f}{\partial x}(a,b) = f_x(a,b)$. We call this the sensitivity coefficient of f with respect to x at (a,b). Similarly $\frac{\partial f}{\partial y}(a,b) = f_y(a,b)$ is the sensitivity coefficient of f with respect to y.

The elasticity of the quantity z relative to x is the percent change in z given a 1% change in x from $x=a$ with no change in $y=b$. We calculate this using the linear approximation of the percent change $\left(\frac{\Delta f}{f(a,b)} \cdot 100\right)$.

Examples:

1. Consider a cylindrical rod with height 100cm and diameter 5cm.
(a) If the measuring instrument has an error of 0.1cm, estimate using linear approximation the corresponding error in the value of the volume if the above measurements are used.

(b) What is the sensitivity of the volume of the given rod to its (i) height and (ii) diameter? That is compute the sensitivity coefficients with respect to the height and diameter at $(100,5)$

(c) Discuss the elasticity of the volume relative to its dimensions. That is discuss how the percentage change in volume for a 1% change in the height compared to the percentage change in volume for a 1% change in the diameter.

Recall: volume of a cylinder, $V = \pi r^2 h$
we are given height and diameter: $V = \pi(\frac{1}{2}d)^2 h = \frac{1}{4}\pi d^2 h$

(a) error in volume given error in height and diameter

$$\Delta V \approx \frac{\partial V}{\partial h}(a,b) \cdot \Delta h + \frac{\partial V}{\partial d}(a,b)\, \Delta d$$

$$= \left(\tfrac{1}{4}\pi d^2\right)(\pm 0.1) + \left(\tfrac{1}{2}\pi d h\right)(\pm 0.1)$$

$$= \tfrac{1}{4}\pi(5)^2(\pm 0.1) + \tfrac{1}{2}\pi(5)(100)(\pm 0.1)$$

$$= \underbrace{\pm 0.625\pi}_{\text{error in height}} \underbrace{\pm 25\pi}_{\text{error in diameter}}$$

(b) sensitivity w.r.t height and diameter i.e. $\frac{\partial V}{\partial h}(100,5)$ and $\frac{dV}{\partial d}(100,5)$

(i) w.r.t. height

$$\frac{\partial V}{\partial h} = \frac{1}{4}\pi d^2$$

$$\frac{\partial V}{\partial h}(100,5) = \frac{1}{4}\pi(5)^2 = \frac{25}{4}\pi$$

(ii) w.r.t diameter

$$\frac{\partial V}{\partial d} = \frac{1}{4}\pi(2d)h = \frac{1}{2}\pi dh$$

$$\frac{\partial V}{\partial d}(100,5) = \frac{1}{2}\pi(5)(100) = 250\pi$$

(c) elasticity w.r.t height and diameter, $\frac{\Delta V}{V(a,b)} \cdot 100$ w/ (i) $\Delta h = 1\%$, $\Delta d = 0$; (ii) $\Delta h = 0$, $\Delta d = 1\%$

note: we want to change one variable and leave the other constant ($\Delta = 0$)
i.e. $\Delta f \approx f_x(a,b)\Delta x + f_y(a,b)\Delta y$ simplies as $\Delta = 0$ for one variable

$$\Delta V \approx V_h(100,5)\Delta h + V_d(100,5)\Delta d$$

$$= \frac{25}{4}\pi \cdot \Delta h + 250\pi \cdot \Delta d$$

(i) w.r.t. height

$$\Delta h = 1\% \text{ of } 100 = (0.01)(100) = 1$$

$$\Delta V = \frac{25}{4}\pi(1) + 250\pi \cdot (0) = \frac{25}{4}\pi$$

$$\% \text{ change} = \frac{\Delta V}{V(a,b)} \cdot 100 = \frac{\frac{25\pi}{4}}{625\pi} \cdot 100 = 1\%$$

(ii) w.r.t. diameter

$$\Delta d = 1\% \text{ of } 5 = (0.01)(5) = 0.05$$

$$\Delta V = \frac{25}{4}\pi(0) + 250\pi\left(\frac{1}{20}\right) = \frac{25}{2}\pi$$

$$\% \text{ change} = \frac{\Delta V}{V(a,b)} \cdot 100 = \frac{\frac{25\pi}{2}}{625\pi} \cdot 100 = 2\%$$

Recap: sensitivity w.r.t. = partial derivative w.r.t

elasticity w.r.t = percent change w/ $\Delta = 1\%$ & 0 respectively
i.e. $\frac{f_x(a,b) \cdot a}{f(a,b)}$ and $\frac{f_y(a,b) \cdot b}{f(a,b)}$