Sequences and Series

Sequences

A sequence is a list of numbers written in a specific order. This list may or may not be infinite, but our class focuses on infinite sequences. A eneral se uence has terms $a_1, a_2, a_3, \dots, a_n, a_{n+1}, \dots$ Let's take a look at a couple of sequences:

example. Write down the first few terms of each of the following sequences:

(a) $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$:

$a_1 = \frac{1+1}{(1)^2} = \frac{2}{1} = 2$, $a_2 = \frac{2+1}{(2)^2} = \frac{3}{4}$, $a_3 = \frac{3+1}{(3)^2} = \frac{4}{9}$, $a_4 = \frac{4+1}{(4)^2} = \frac{5}{16}$, $a_5 = \frac{5+1}{(5)^2} = \frac{6}{25}$

(b) $\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$:

$a_0 = \frac{(-1)^{0+1}}{2^0} = \frac{-1}{1} = -1$, $a_1 = \frac{(-1)^{1+1}}{2^1} = \frac{1}{2}$, $a_2 = \frac{(-1)^{2+1}}{2^2} = \frac{-1}{4}$, $a_3 = \frac{(-1)^{3+1}}{2^3} = \frac{1}{8}$, $a_4 = \frac{(-1)^{4+1}}{2^4} = \frac{-1}{16}$

(c) $\{b_n\}_{n=1}^{\infty}$ where b_n is the n^{th} digit of π :

$a_1 = 3$, $a_2 = 1$, $a_3 = 4$, $a_4 = 1$, $a_5 = 5$

We can also use pattern recognition to solve for the general term of the sequence and the limit of the sequence as $n \to \infty$.

example. For each sequence below, write down (a) the formula for the general term of the sequence and (b) the limit of the sequence:

1. $\frac{2}{4\cdot 5}, \frac{2}{5\cdot 6}, \frac{2}{6\cdot 7}, \frac{2}{7\cdot 8}, \dots$

(a). patterns I see:
- top is always 2
- bottom is a # times #+1

$a_n = \frac{2}{(n+3(n+4)}$

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2}{(n+3)(n+4)}$

$= \lim_{n\to\infty} \frac{2}{n^2+7n+12}$

$= 0$

2. $2, -2, 2, -2, .$

(a) patterns I see:
- odd terms 2
- even terms -2

i.e $a_{n+1} = -1 \cdot a_n$

$a_n = (-1)^{n+1} \cdot 2$

(b) $\lim_{n\to\infty} a_n =$ D.N.E.

due to never converging

Summing an Infinite Series

Start with a sequence $\{a_n\}_{n=1}^{\infty}$, we define a new sequence of "partial sums", $\{s_n\}_{n=1}^{\infty}$, by the following: $s_1=a_1$, $s_2=a_1+a_2$, $s_3=a_1+a_2+a_3$, $s_4=a_1+a_2+a_3+a_4$, ..., $s_n=a_1+a_2+\dots+a_n=\sum_{i=1}^{n} a_i$.

Recall that $\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_n$. We say that the infinite series converges if the sequence $\{s_n\}_{i=1}^{\infty}$ is convergent and its limit is finite. Similarly, if the sequence of partial sums is divergent then so is the series.

example. Determine if the following series are convergent or divergent. If it converges, determine its value.

(a) $\sum_{n=1}^{\infty} n$

$\{a_n\} = \{1,2,3,4,5,\dots\}$

$\{s_n\} = \{1,3,6,10,15,\dots\} = \left\{\frac{n(n+1)}{2}\right\}_{n=1}^{\infty}$

divergent

(b) $\sum_{n=3}^{\infty} (\sqrt{n+1}-\sqrt{n})$

$\{a_n\}_{n=3}^{\infty} = \{\sqrt{4}-\sqrt{3}, \sqrt{5}-\sqrt{4}, \sqrt{6}-\sqrt{5}, \sqrt{7}-\sqrt{6}, \dots\}$

$S_1 = \sqrt{4}-\sqrt{3}$

$S_2 = (\sqrt{4}-\sqrt{3})+(\sqrt{5}-\sqrt{4})$
$= -\sqrt{2}+\sqrt{3}$

$S_3 = \sqrt{4}-\sqrt{3}+\sqrt{5}-\sqrt{4}+\sqrt{6}-\sqrt{5}$
cancel cancel
$= -\sqrt{2}+\sqrt{3}$

$\{s_n\}_{n=3}^{\infty} = \{\sqrt{4}-\sqrt{3}, -\sqrt{3}+\sqrt{5}, -\sqrt{3}+\sqrt{6}, \dots\}$

convergent to $-\sqrt{3}$
due to this cancellation

note the index: $s_1 = a_3$

$s_2 = a_3+a_4$

$s_3 = a_3+a_4+a_5$

...

$S_N = a_3+\dots+a_{N+2}$

$= -\sqrt{2}+\sqrt{(N+2)+1}$