Special Series

Geometric Series

A geometric series is any series that can be written in the form

$$\sum_{n=1}^{\infty} ar^{n-1},$$

or with an index shift

$$\sum_{n=0}^{\infty} ar^{n}.$$

The partial sums are

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

The series will converge provided the partial sums form a convergent sequence,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{a}{1-r} - \frac{ar^n}{1-r}\right) = \lim_{n\to\infty}\frac{a}{1-r} - \lim_{n\to\infty}\frac{ar^n}{1-r} = \frac{a}{1-r} - \frac{a}{1-r}\cdot\lim_{n\to\infty} r^n.$$

The limit will exist and be finite provided $-1<r<1$, in fact $\lim_{n\to\infty} r^n = 0$ when $-1\le r\le 1$.

Therefore, a geometric series will converge when $|r|<1$ to $\lim_{n\to\infty} S_n = \frac{a}{1-r}$.

Example:

1. Determine if the following series converge or diverge. If they converge give the value of the series.

(a) $\sum_{n=1}^{\infty} 9^{-n+2}\,4^{n+1}$

$= \sum_{n=1}^{\infty} 9^{-(n-2)}\,4^{n+1}$

$= \sum_{n=1}^{\infty} \frac{4^{n-1}4^2}{9^{n-2}}$

$= \sum_{n=1}^{\infty} \frac{4^{n-1}4^2}{9^{n-1}9^{-1}}$

$= \sum_{n=1}^{\infty} 144\left(\frac{4}{9}\right)^{n-1}$

$a=144$, $r=\frac{4}{9}<1$

$\frac{a}{1-r} = \frac{144}{1-4/9}$

$= \frac{9}{5}(144)$

$= \frac{1296}{5}$

(b) $\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$

$= \sum_{n=0}^{\infty} \frac{((-4)^3)^n}{5^n 5^{-1}}$

$= \sum_{n=0}^{\infty} \frac{5(-64)^n}{5^n}$

$= \sum_{n=0}^{\infty} 5\left(\frac{-64}{5}\right)^n$

$a=5$, $r=-\frac{64}{5}>1$

series diverges since $r>1$

(c) $\sum_{n=1}^{\infty} 5\cdot\left(\frac{1}{10}\right)^n$

$a=5$, $r=\frac{1}{10}<1$

$\frac{a}{1-r} = 5\cdot\frac{1}{1-\frac{1}{10}}$

note something about S_N:

$S_1 = 5\cdot\left(\frac{1}{10}\right) = 0.5$

$S_2 = 5\cdot\left(\frac{1}{10}\right) + 5\cdot\left(\frac{1}{10}\right)^2 = 0.5 + 0.05$

$\vdots$

$S_N = 0.5 + 0.05 + 0.005 + \ldots + 5\left(\frac{1}{10}\right)^N$

$= 0.555 + \ldots + 5\cdot\left(\frac{1}{10}\right)^N$

$\lim_{n\to\infty} S_n = 0.\overline{55}$

Telescoping Series

A telescoping series is best shown through an example.

Example:

2. Determine if the following series converges or diverges. If it converges find its value.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2}$

$$S_n = \sum_{i=0}^{n} \frac{1}{i^2+3i+2}$$

$$= \sum_{i=0}^{n} \frac{1}{(i+2)(i+1)}$$

$$= \sum_{i=0}^{n} \left(\frac{1}{i+1} - \frac{1}{i+2}\right)$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

assumed

as $n \to \infty$ this cancels too

$$= 1 - \frac{1}{n+2}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+2}\right)$$

$$= 1$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3}$

$$S_n = \sum_{i=1}^{n} \left(\frac{1/2}{i+1} - \frac{1/2}{i+3}\right)$$

$$= \frac{1}{2} \sum_{i=1}^{n} \left(\frac{1}{i+1} - \frac{1}{i+3}\right)$$

$$= \frac{1}{2}\left[\left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \ldots + \overset{n-2}{\left(\frac{1}{n-1} - \frac{1}{n+1}\right)} + \overset{n-1}{\left(\frac{1}{n} - \frac{1}{n+2}\right)} + \overset{n}{\left(\frac{1}{n+1} - \frac{1}{n+3}\right)}\right]$$

takes 2 terms to cancel

assumed cancel

survived

$$= \frac{1}{2}\left[\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\right]$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}\left(\frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$

$$= \frac{5}{12}$$

It is not always obvious if a series is telescoping, you see it when you look at the partial sum. Not all partial fractions are telescoping, take $\displaystyle\sum_{n=1}^{\infty} \frac{3+2n}{n^2+3n+2} = \sum_{n=1}^{\infty}\left(\frac{1}{n+1} + \frac{1}{n+2}\right)$.