Geometric Series Example

Example:

1 A drug is designed so that 60% remains in the body at the end of each 24 hour period (one day). If 30 mg of the drug is given daily to a patient find (A) the amount of the drug in the body after 10 days before the next dose is given, (B) the approximate amount of the drug in the body after a very long time assuming measurement is done before the next dose is given.

Medicine is given at noon every day. We will do our measurements at 11:59 am.

measurement 1: (dose 1)(% left after 24hrs) = (30)(0.6)

measurement 2: (dose 1)(% left after 1st 24hrs)(% left after 2nd 24hrs) + (dose 2)(% left after 24 hrs)

$= (30)(0.6)(0.6) + (30)(0.6)$

measurement 3: $(\text{dose }1)(\%\text{ left})^{3\text{days}} + (\text{dose }2)(\%\text{ left})^{2\text{days}} + (\text{dose }3)(\%\text{ left})^{1\text{ day}}$

$= (30)(0.6)^3 + (30)(0.6)^2 + (30)(0.6)^1$

$\vdots$

measurement 10 $= \sum_{n=1}^{10} (30)(0.6)^n$

$\vdots$

(A) amount at 11:59am after 10 days

$$S_{10} = \underbrace{\sum_{n=1}^{10} (30)(0.6)^n}_{} = \frac{a(1-r^{10})}{1-r} = \frac{(30)(0.6)(1-0.6^{10})}{1-0.6} = \frac{18(1-(0.6)^{10})}{0.4} = 45(1-(0.6)^{10})$$

geometric: $a = (30)(0.6)^1$, $r = 0.6$

(B) amount at 11:59am after <u>A LOT</u> of days

Since $|r| \leq 1$, the geometric series converges

$$\lim_{n\to\infty} S_n = \frac{a}{1-r} = \frac{18}{1-0.6} = 45\text{mg}$$ more than a dose!

Convergence of Series

Ratio Test

Suppose we have the series Σa_n. Define $L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. Then

(i) if $L<1$ the series is convergent

(ii) if $L>1$ the series is divergent

(iii) if $L=1$ the test fails

Examples:

1. Determine if the following series are convergent or divergent.

(a) $\sum_{n=1}^{\infty}\frac{(-10)^n}{4^{2n+1}(n+1)}$

$$a_n=\frac{(-10)^n}{4^{2n+1}(n+1)}$$

$$a_{n+1}=\frac{(-10)^{n+1}}{4^{2(n+1)+1}((n+1)+1)}$$

$$=\frac{(-10)^{n+1}}{4^{2n+3}(n+2)}$$

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(-10)^{n+1}}{4^{2n+3}(n+2)}\cdot\frac{4^{2n+1}(n+1)}{(-10)^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{-10(n+1)}{4^2(n+2)}\right|$$

$$=\frac{10}{16}\lim_{n\to\infty}\frac{n+1}{n+2}$$

$$=\frac{10}{16}$$

converges as $L<1$

(b) $\sum_{n=0}^{\infty}\frac{n!}{5^n}$

$$a_n=\frac{n!}{5^n}$$

$$a_{n+1}=\frac{(n+1)!}{5^{n+1}}$$

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(n+1)!}{5^{n+1}}\cdot\frac{5^n}{n!}\right|$$

$$=\lim_{n\to\infty}\left|\frac{1}{5}\cdot\frac{(n+1)n!}{n!}\right|$$

$$=\lim_{n\to\infty}\frac{n+1}{5}$$

$$=\infty$$

diverges

Power Series

A power series about a is any series that can be written in the form $\sum_{n=0}^{\infty}a_n(x-c)^n$ where c and a_n are numbers and the a_n's are called the coefficients of the series. The same principles for converges apply, but now we have a variable that will affect when the series converges. There exists a number R such that the power series converges for $|x-a|<R$ and will diverge for $|x-a|>R$. This number is called the radius of convergence for the series.

Examples:

2. Determine the radius of convergence for the following power series.

(a) $\sum_{k=1}^{\infty} \frac{(x-2)^k}{k^3}$

ratio test

$\lim_{k\to\infty} \left| \frac{(x-2)^{k+1}}{(k+1)^3} \cdot \frac{k^3}{(x-2)^k} \right|$

$= \lim_{k\to\infty} \left| (x-2) \cdot \frac{k^3}{(k+1)^3} \right|$

$= |x-2| \lim_{k\to\infty} \left(\frac{k}{k+1} \right)^3$

$= |x-2| < 1$

$x \in (1,3)$
radius $= 1$

(b) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

ratio test

$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$

$= \lim_{n\to\infty} \left| \frac{x}{n+1} \right|$

$= x \lim_{n\to\infty} \left| \frac{1}{n+1} \right|$

$= x \cdot (0)$

$= 0 < 1$

$x \in (-\infty, \infty)$
radius $= \infty$

(c) $\sum_{k=1}^{\infty} \frac{x^{2k}}{2k+1}$

ratio test

$\lim_{k\to\infty} \left| \frac{x^{2(k+1)}}{2(k+1)+1} \cdot \frac{2k+1}{x^{2k}} \right|$

$= \lim_{k\to\infty} \left| \frac{x^{2k+2}}{2k+3} \cdot \frac{2k+1}{x^{2k}} \right|$

$= \lim_{k\to\infty} \left| (x^2) \cdot \frac{2k+1}{2k+3} \right|$

$= |x^2| \cdot \lim_{k\to\infty} \left(\frac{2k+1}{2k+3} \right)$

$= |x^2| < 1$

$x \in (-1, 1)$
radius $= 1$

Exit Ticket Special Series

Geometric Series A geometric series is any series that can be written in the form

$$\sum_{n=1}^{\infty} a \cdot r^{n-1}.$$

The partial sums are

$$s_n = \frac{a(1-r^n)}{1-r}.$$

When $|r| \leq 1$, the geometric series converges to

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}$$

Determine if the following sequences are geometric or telescoping. Find s_5. Determine if the series converges or diverges. If the series converges, state what it converges to.

1. $\sum_{n=0}^{\infty} 3^{2+n}2^{1-3n}$ Geometric

$= \sum_{n=0}^{\infty} 3^2 \cdot 3^n \cdot 2^1 \cdot 2^{-3n}$

$= \sum_{n=0}^{\infty} 18 \cdot \frac{3^n}{2^{3n}}$

$= \sum_{n=0}^{\infty} 18 \cdot \left(\frac{3}{8}\right)^n$

$= \sum_{n=1}^{\infty} 18 \cdot \left(\frac{3}{8}\right)^{n-1}$

$a = 18 \quad r = \frac{3}{8} \leq 1$

$S_5 = \frac{a(1-r^n)}{(1-r)}$

$= \frac{18(1-(3/8)^5)}{1-3/8}$

$= \frac{18(1-(3/8)^5)}{5/8}$

$\lim_{n\to\infty} S_n = \frac{a}{1-r}$

$= \frac{18}{1-3/8}$

$= \frac{18}{5/8}$

2. $\sum_{n=1}^{\infty} \frac{5}{6n}$ neither

Needs ratio test

$S_5 = \frac{5}{6} + \frac{5}{12} + \frac{5}{18} + \frac{5}{24} + \frac{5}{30}$

3. $\sum_{n=2}^{\infty} \frac{5^{n+1}}{7^{n-2}}$ Geometric

index shift

$= \sum_{n=1}^{\infty} \frac{5^{(n+1)+1}}{7^{(n+1)-2}}$

$= \sum_{n=1}^{\infty} \frac{5^{n+2}}{7^{n-1}}$

$= \sum_{n=1}^{\infty} \frac{5^2}{7^{-1}} \cdot \frac{5^n}{7^n}$

$= \sum_{n=1}^{\infty} 5^2 \cdot 7 \cdot \left(\frac{5}{7}\right)^n$

$a = 5^2 \cdot 7 = 175, \; r = \frac{5}{7} \leq 1$

$S_5 = \frac{a(1-r^n)}{1-r}$

$= \frac{175(1-(5/7)^5)}{1-5/7}$

$= \frac{175(1-(5/7)^5)}{2/7}$

$\lim_{n\to\infty} S_n = \frac{175}{1-5/7}$

$= \frac{175}{2/7}$

partial fractions

4. $\sum_{n=4}^{\infty} \boxed{\frac{10}{n^2-4n+3}}$ Telescoping

$= \sum_{n=4}^{\infty} \frac{5}{n-3} - \frac{5}{n-1}$

$S_5 = \left[\frac{5}{1} - \frac{5}{3}\right] + \left[\frac{5}{2} - \frac{5}{4}\right] + \left[\frac{5}{3} - \frac{5}{5}\right] + \left[\frac{5}{4} - \frac{5}{6}\right] + \left[\frac{5}{5} - \frac{5}{7}\right]$

$= \frac{5}{1} + \frac{5}{2} - \frac{5}{6} - \frac{5}{7}$

$\lim_{n\to\infty} S_n = \left[\frac{5}{1} - \frac{5}{3}\right] + \left[\frac{5}{2} - \frac{5}{4}\right] + \left[\frac{5}{3} - \frac{5}{5}\right] + \left[\frac{5}{4} - \frac{5}{6}\right] + \ldots$

$= 5 + \frac{5}{2}$

$= \frac{15}{2}$