10360 Exam 03 Review Set01 Name ______________________________

1. Solve the following initial value problem:

$$(x^2+1)y' = 2xy - 3(x^2+1); \qquad y(1) = -1.$$

$y' + A(x)\cdot y = B(x)$

$y' = \frac{2x}{x^2+1}\cdot y - 3$

$y' + \frac{-2x}{x^2+1}\cdot y = -3$

Step 1: $A(x) = \frac{-2x}{x^2+1}$; $B(x) = -3$

Step 2: $\alpha(x) = e^{\int A(x)dx}$

$= e^{\int \frac{-2x}{x^2+1}dx}$

$= e^{-\ln|x^2+1|}$

$= e^{\ln|(x^2+1)^{-1}|}$

$= (x^2+1)^{-1}$

$= \frac{1}{x^2+1}$

Step3: $y(x) = \frac{1}{\alpha(x)}\left[\int \alpha(x)\cdot B(x)dx + C\right]$

$= (x^2+1)\left[\int \frac{1}{x^2+1}\cdot(-3)dx + C\right]$

$y(x) = (x^2+1)\left[-3\arctan(x) + C\right]$

Step4: @ $(1,-1)$

$y(1) = ((1)^2+1)\left[-3\arctan(1) + C\right]$

$-1 = 2\left[-3(\pi/4) + C\right]$

$-1 = -3\pi/2 + 2C$

$-1 + 3\pi/2 = 2C$

$\frac{-2+3\pi}{2} = 2C$

$-\frac{2+3\pi}{4} = C$

$y(x) = (x^2+1)\left[-3\arctan(x) + \frac{3\pi-2}{4}\right]$

10360 Exam 03 Review Set02 Name ______________________

2a. A function $f(x, y)$ is given by the selected points in the table. Using linear approximation at $(2.5, -1)$, estimate $f(2.6, -0.8)$.

(a,b) (x, y)

		(x)		
		1.5	**2**	**2.5**
(y)	**1**	3	1.5	-1
	0	4	3.5	2
	−1	3	2.5	0.5

$f(x,y) \approx f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$

$= -4(2.6-2.5) + \frac{3}{2}(-0.8-(-1)) + 0.5$

$= -4(0.1) + \frac{3}{2}(0.2) + 0.5$

$= -0.4 + 0.3 + 0.5$

$= 0.4$

$f_x \approx \frac{0.5-2.5}{0.5} = \frac{-2}{1/2} = -4$

backwards

$f_y \approx \frac{2-0.5}{1} = \frac{3/2}{1} = \frac{3}{2}$

forward

2b. Using central difference formula, estimate the sensitivity of f relative to x at $(2, 1)$.

central $= \frac{f(2.5,1) - f(1.5,1)}{2.5 - 1.5}$

$= \frac{-1-3}{1}$

$= -4$

2c. Estimate the elasticity of f relative to y at $(2, 1)$.

$\frac{\Delta f}{f(a,b)} \cdot 100$ $\Delta x = 0$
$\Delta y = 1\%$ of 1

$f_y(2,1) \approx \frac{3.5-1.5}{-1} = -2$

$\Delta f = f_x(a,b)\Delta x + f_y(a,b)\Delta y$

$= 0 + (-2)(\frac{1}{100})$

$= -\frac{1}{50}$

elasticity $= \frac{-1/50}{1.5} \cdot 100$

$= \frac{-2}{3/2}$

$= -4/3$

3a. A single dosage of 100mg of Drug A is given to a patient at 8:00am each day. 10% of the drug remains in the body after one full day period (8:00am next day). What is the amount of drug in the body **20 days** after the treatment started before the next dose is given at 8:00am?

$S_1 = \underbrace{100\cdot(0.1)}_{\text{dose 1}}$ = amount of drug in body at 7:59am

$$S_2 = \underbrace{100\cdot(0.1)^2}_{\text{dose 1}} + \underbrace{100\ (0.1)^1}_{\text{dose 2}}$$

$$S_{20} = a_1 + \ldots + a_{20}$$

$$= 100\cdot(0.1)^1 + \ldots + 100\cdot(0.1)^{20}$$

$$= \sum_{n=1}^{20} \underbrace{100\,(0.1)}_{a}\cdot\underbrace{(0.1)}_{r}{}^{n-1}$$

$$S_{20} = \frac{a(1-r^{20})}{1-r}$$

$$= \frac{100(0.1)\,(1-(0.1)^{20})}{1-0.1}$$

$$= \frac{100}{9}\,(1-(0.1)^{20})$$

3b. Using geometric series, write the following repeated decimal as a fraction.

$21.\overline{02} = 21 + 0.02 + 0.0002 + 0.000002 + \ldots$

$$= 21 + \sum_{n=1}^{\infty} \underbrace{0.02}_{a}\cdot\underbrace{\left(\tfrac{1}{100}\right)}_{r}{}^{n-1}$$

$$= 21 + \frac{0.02}{1-1/100}$$

$$= 21 + \frac{2/100}{99/100}$$

$$= 21 + \frac{2}{99}$$

$$= \frac{2079+2}{99}$$

$$= \frac{2081}{99}$$

3c. Consider the geometric series $\sum_{n=2}^{\infty} 5\left(\frac{2}{3}\right)^n$. Find the number of terms N we need to add so that the Nth partial sum has value $\frac{20}{3} - 10\left(\frac{2^{21}}{3^{21}}\right)$.

$$\sum_{n=2}^{\infty} 5\left(\tfrac{2}{3}\right)^n = \sum_{n=1}^{\infty} 5\left(\tfrac{2}{3}\right)^{n+1} = \sum_{n=1}^{\infty} 5\left(\tfrac{2}{3}\right)^2 \cdot \left(\tfrac{2}{3}\right)^{n-1}$$

$$S_N = \frac{a(1-r^N)}{1-r} = \frac{20}{3} - 10\left(\frac{2^{21}}{3^{21}}\right)$$

$$\frac{5\left(\frac{2}{3}\right)^2\left(1-(2/3)^N\right)}{1-2/3} = \frac{20}{3} - 10\left(\frac{2^{21}}{3^{21}}\right)$$

$$\frac{20/9\left(1-(2/3)^N\right)}{1/3} = \frac{20}{3} - 10\left(\frac{2^{21}}{3^{21}}\right)$$

$$\frac{20}{3}\left(1-(2/3)^N\right) = \frac{20}{3} - 10\left(\frac{2^{21}}{3^{21}}\right)$$

$$\frac{20}{3} - \frac{20}{3}\left(\frac{2^N}{3^N}\right) = \frac{20}{3} - 10\left(\frac{2^{21}}{3^{21}}\right)$$

$$-\frac{20}{3}\left(\frac{2^N}{3^N}\right) = -10\left(\frac{2^{21}}{3^{21}}\right)$$

$$\frac{2}{3}\left(\frac{2^N}{3^N}\right) = \frac{2^{21}}{3^{21}}$$

$$\frac{2^N}{3^N} = \frac{2^{20}}{3^{20}}$$

$$N = 20$$

4. Consider a tank that contains 100 liters of brine with concentration 0.1 kg/L. Brine of concentration 0.5 kg/L is **pumped in** at a certain rate of R L/min syrup while mixture is continuously stirred. If the mixture is **pumped out** at the same rate and the concentration of the mixture is found to be 0.3 kg/L after 30 minutes, what is the value of the flow rate R in L/min.

$y(t)$ = amount of salt

$v(t)$ = volume of tank $= 100 + R - R$
$= 100$

$\frac{dy}{dt} =$ amount of salt in $-$ amount of salt out

$=$ (rate in)(concentration in) $-$ (rate out)(concentration out)

$= (R)(0.5) - (R)\left(\frac{y(t)}{v(t)}\right)$

$$\frac{dy}{dt} = 0.5R - \frac{R}{100}\cdot y$$

$$\frac{dy}{dt} = \frac{1}{2}R - \frac{R}{100}y$$

$$\frac{dy}{dt} = \frac{1}{100}R(50-y)$$

$\leftarrow$ separable $\frac{dy}{dt} = P(x)\cdot Q(y)$

$$\int \frac{1}{50-y}\,dy = \int \frac{1}{100}R\,dt$$

$$-\ln|50-y| = \frac{1}{100}R\cdot t + c$$

@ $y(0) = 10$

$$-\ln|50-10| = \frac{1}{100}R\cdot(0) + c$$

$$-\ln|40| = c$$

@ $y(30) = 30$

$$-\ln|50-30| = \frac{1}{100}R(30) - \ln|40|$$

$$-\ln|20| = \frac{3}{10}R - \ln|40|$$

$$\ln|40| - \ln|20| = \frac{3}{10}R$$

$$\ln\left|\frac{40}{20}\right| = \frac{3}{10}R$$

$$\ln|2| = \frac{3}{10}R$$

$$\frac{10}{3}\ln|2| = R$$

concentration $= 0.1$ at $t = 0$

concentration $= \frac{\text{salt}}{\text{volume}}$

$0.1 = \frac{y(0)}{100}$

$10 = y(0)$

concentration $= 0.3$ at $t = 30$

concentration $= \frac{\text{salt}}{\text{volume}}$

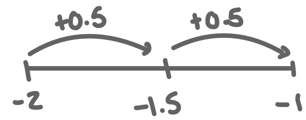
$0.3 = \frac{y(30)}{100}$

$30 = y(30)$

5. Consider the initial value problem:

$$y' = y(y+t); \qquad y(-2) = 1$$

Use Euler's method with two equal steps to estimate $y(-1)$.



$x_0 = (-2)$

$x_1 = -2 + 0.5$
$= -1.5$

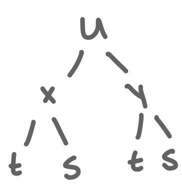
$x_2 = -1$

$y_0 = 1$

$y_1 = 1 + \frac{1}{2}\left((1)((1) + (-2))\right)$
$= 1 + \frac{1}{2}(-1)$
$= 1 - \frac{1}{2}$
$= \frac{1}{2}$

$y_2 = \frac{1}{2} + \frac{1}{2}\left((\tfrac{1}{2})(\tfrac{1}{2} + (-1.5))\right)$
$= \frac{1}{2} + \frac{1}{2}\left((\tfrac{1}{2})(-1)\right)$
$= \frac{1}{2} + \frac{1}{2}(-\frac{1}{2})$
$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

6. Find $\dfrac{\partial u}{\partial t}$ when $t = 1$ and $s = -1$ where $u = e^{x+2y}$; $x = t + 2s$, and $y = 3t - s$. Find also a formula for $\dfrac{\partial u}{\partial s}$ in terms of s and t.



$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t}$

$= [e^{x+2y}][1] + [2e^{x+2y}][3]$

$= e^{x+2y} + 6e^{x+2y}$

$= 7e^{x+2y}$

$= 7e^{(t+2s)+2(3t-s)}$

$= 7e^{t+2s+6t-2s}$

$= 7e^{7t}$

@ $t=1$, $s=-1$: $\frac{\partial u}{\partial t} = 7e^{7(1)} = 7e^{7}$

$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$

$= [e^{x+2y}][2] + [2e^{x+2y}][-1]$

$= 2e^{x+2y} - 2e^{x+2y}$

$= 0$

7. Using implicit differentiation, find $\frac{\partial z}{\partial y}(-1,0,0)$ if $\cos(x^2yz) + 5e^z = yz^2 - 3xy^2 + 6$.

product product

$$-\sin(x^2yz)\left[(x^2)z + \frac{\partial z}{\partial y}\cdot x^2y\right] + 5e^z\cdot\frac{\partial z}{\partial y} = \left[(1)(z^2) + \left(2z\cdot\frac{\partial z}{\partial y}\right)(y)\right] - 6xy + 0$$

$$-\sin(x^2yz)\left[x^2z + x^2y\cdot\frac{\partial z}{\partial y}\right] + 5e^z\cdot\frac{\partial z}{\partial y} = z^2 + 2yz\cdot\frac{\partial z}{\partial y} - 6xy$$

$$-x^2z\cdot\sin(x^2yz) - x^2y\cdot\sin(x^2yz)\cdot\frac{\partial z}{\partial y} + 5e^z\cdot\frac{\partial z}{\partial y} = z^2 + 2yz\cdot\frac{\partial z}{\partial y} - 6xy$$

$$-x^2y\cdot\sin(x^2yz)\cdot\frac{\partial z}{\partial y} + 5e^z\cdot\frac{\partial z}{\partial y} - 2yz\cdot\frac{\partial z}{\partial y} = z^2 - 6xy + x^2z\cdot\sin(x^2yz)$$

$$\frac{\partial z}{\partial y}\left[-x^2y\cdot\sin(x^2yz) + 5e^z - 2yz\right] = z^2 - 6xy + x^2z\cdot\sin(x^2yz)$$

$$\frac{\partial z}{\partial y} = \frac{z^2 - 6xy + x^2z\cdot\sin(x^2yz)}{5e^z - 2yz - x^2y\cdot\sin(x^2yz)}$$

@$(-1,0,0)$

$$\frac{\partial z}{\partial y} = \frac{(0)^2 - 6(-1)(0) - (-1)^2(0)\sin((-1)^2(0)(0))}{5e^0 - 2(0)(0) + (-1)^2(0)\sin((-1)^2(0)(0))}$$

$$= \frac{0 - 0 - 0}{5(1) - 0 + 0}$$

$$= 0$$

8. Consider the function $f(x, y) = \ln(x^2 y) + 10$

8a. Use linear approximation to estimate the percentage change of f when (x, y) changes from $(-1, 1)$ to $(-0.9, 1.1)$

(a,b)

percent change $= \frac{\Delta f}{f(a,b)} \cdot 100$

$\Delta f = f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$f_x = \frac{1}{x^2y} \cdot 2x$

$f_x(-1,1) = \frac{2(-1)}{(-1)^2(1)} = \frac{-2}{1} = -2$

$f_y = \frac{1}{x^2y} \cdot 1$

$f_y(-1,1) = \frac{1}{(-1)^2(1)} = \frac{1}{1} = 1$

$\Delta f = -2(-0.9-(-1)) + 1(1.1-1)$

$= -2(0.1) + 1(0.1)$

$= -0.2 + 0.1$

$= -0.1$

$f(-1,1) = \ln((-1)^2(1)) + 10$

$= \ln(1) + 10$

$= 0 + 10$

$= 10$

percent change $= \frac{-0.1}{10} \cdot 100$

$= -1$

8b. Use the linear approximation of $f(x, y)$ at $(-1, 1)$ to estimate the value of $f(-0.9, 1.1)$.

$\Delta f = f(x,y) - f(a,b)$

$f(x,y) \approx \Delta f + f(a,b)$

$= -0.1 + 10$

$= 9.9$

9a. Find the 53rd partial sum of the series $\sum_{n=3}^{\infty} \frac{2}{4n^2+8n+3}$.

$S_{53} = a_3 + \ldots + a_{N+2}$

$= a_3 + \ldots + a_{55}$

$$\frac{2}{4n^2+8n+3} = \frac{2}{(2n+1)(2n+3)} = \frac{A}{(2n+1)} + \frac{B}{(2n+3)}$$

$2 = A(2n+3) + B(2n+1)$

$n = -1/2: \ 2 = A(2(-1/2)+3) + B(2(-1/2)+1)$

$2 = A(-1+3) + B(-1+1)$

$2 = A(2) + 0$

$1 = A$

$n = -3/2: \ 2 = A(2(-3/2)+3) + B(2(-3/2)+1)$

$2 = A(-3+3) + B(-3+1)$

$2 = 0 + B(-2)$

$-1 = B$

$$\sum_{n=3}^{\infty} \frac{1}{2n+1} + \frac{-1}{2n+3}$$

$$S_{53} = \underbrace{\left[\frac{1}{7} - \frac{1}{9}\right]}_{a_3} + \underbrace{\left[\frac{1}{9} - \frac{1}{11}\right]}_{a_4} + \underbrace{\left[\frac{1}{11} - \frac{1}{13}\right]}_{a_5} + \ldots + \underbrace{\left[\frac{1}{109} - \frac{1}{111}\right]}_{a_{54}} + \underbrace{\left[\frac{1}{111} - \frac{1}{113}\right]}_{a_{55}}$$

$$= \frac{1}{7} - \frac{1}{113}$$

9b. Find the sum of the series $\sum_{n=3}^{\infty} \frac{2}{4n^2 + 8n + 3}$.

$\frac{1}{7}$ as all else eventually cancels