Taylor Polynomials

Examples:

1a. Find the 3rd-degree Taylor polynomial of $y(t)$ centered at zero, where $y(t)$ is the solution of the initial value probem $y' = y^2 + ty$, $y(0) = 1$.

b. Use your result to estimate $y(0.3)$.

$$T_3(x) = \sum_{n=0}^{3} \frac{f^n(c)}{n!}(x-c)^n = \frac{f(0)}{0!}(x-0)^0 + \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$= y(0) + y'(0)\cdot(x) + \tfrac{1}{2}y''(0)\cdot(x)^2 + \tfrac{1}{6}y'''(0)\cdot(x)^3$$

$y(x) = ??$ unknown — $y(0) = 1$

$y'(x) = y^2 + \underbrace{ty}_{\text{product}}$ given — $y'(0) = (1)^2 + (0)\cdot(1) = 1$

$y''(x) = 2y\cdot y' + (1)(y) + (y')\cdot(t)$ implicit diff.

$= \underbrace{2y\cdot y'}_{\text{product}} + y + \underbrace{ty'}_{\text{product}}$

$y''(0) = 2(1)(1) + (1) + (0)(1)$
$= 2 + 1 = 3$

$y'''(x) = (2y')\cdot y' + y''(2y) + y' + (1)(y') + (y'')\cdot(t)$

$= 2\underbrace{y'\cdot y'}_{\text{not } y^2} + 2y\cdot y'' + 2y' + t\cdot y''$

$y'''(0) = 2(1)(1) + 2(1)(3) + 2(1) + (0)(1)$
$= 2 + 6 + 2 = 10$

$T_3(x) = y(0) + y'(0)\cdot(x) + \tfrac{1}{2}y''(0)\cdot(x)^2 + \tfrac{1}{6}y'''(0)\cdot(x)^3$
$= 1 + 1x + \tfrac{1}{2}(3)x^2 + \tfrac{1}{6}(10)x^3$
$= 1 + x + \tfrac{3}{2}x^2 + \tfrac{5}{3}x^3$

$y(0.3) \approx T_3(0.3) = 1 + (0.3) + \tfrac{3}{2}(0.3)^2 + \tfrac{5}{3}(0.3)^3$

2. The 3rd Macluarin Polynomial of $f(x)$ is given by $T_3(x) = 1 - x^2 + 4x^3$

(a) Find the values $f(0)$, $f'(0)$, $f''(0)$, and $f'''(0)$.

$T_3(x) = \sum_{n=0}^{3} \frac{f^n(0)}{n!}(x)^n$ degree of x = degree of derivative

$\frac{f(0)}{0!} = 1$ — $\frac{f'(0)}{1!}x = 0\cdot x$ — $\frac{f''(0)}{2!}x^2 = -1$ — $\frac{f'''(0)}{3!}x^3 = 4x^3$

$f(0) = 1$ — $f'(0) = 0$ — $f''(0) = -1\cdot 2 = -2$ — $f'''(0) = 4\cdot 3! = 4\cdot 6 = 24$

Lagrange Multipliers

Recall from Calculus A, a continuous function $y=f(x)$ on a closed and bounded interval $a \le x \le b$:

(i) $f(x)$ must attain its minimum and maximum for some values of x in $[a,b]$

(ii) the extrema of $f(x)$ can occur at the endpoints $(x=a,b)$ or at critical points $(f'(x)=0)$ in $a<x<b$.

We say that we are maximizing $y=f(x)$ on the constraint $a \le x \le b$. In higher dimensions, the constraint is no longer an interval.

Lagrange Multipliers

For example, if someone is hiking on a trail their height may vary based on their position (x,y). If we are looking at a topography map (with rings), which has a height function $z=h(x,y)$, then we can find the hikers maximum elevation by maximizing the height by the constraint $g(x,y)=0$ (the hikers path). The critical points are the solution to the following system of equations:

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = \lambda \frac{\partial g}{\partial x}(x,y) \\ \frac{\partial f}{\partial y}(x,y) = \lambda \frac{\partial g}{\partial y}(x,y) \\ g(x,y)=0 \end{cases}$$

Examples:

1a. The height of the slanted roof of a house is given by $h(x,y)=2x+4y+20$. A spider is observed from the top view crawling on the closed path $x^2+y^2=4$. What is the minimum and maximum height attained by the spider?

optimize $h(x,y)=2x+4y+20$ constraint $x^2+y^2-4=0$

$$\begin{cases} h_x = \lambda g_x \\ h_y = \lambda g_y \\ g(x,y)=0 \end{cases} \quad \begin{cases} 2=\lambda\cdot 2x \\ 4=\lambda\cdot 2y \\ x^2+y^2-4=0 \end{cases} \quad \begin{cases} 1=\lambda x \\ 2=\lambda y \\ x^2+y^2-4=0 \end{cases} \quad \begin{cases} x=1/\lambda \\ y=2/\lambda \\ (1/\lambda)^2+(2/\lambda)^2-4=0 \end{cases}$$

$$\frac{1}{\lambda^2}+\frac{4}{\lambda^2}-4=0$$
$$\frac{5}{\lambda^2}=4$$
$$\lambda^2=\frac{5}{4}$$
$$\lambda=\pm\sqrt{5}/2$$

when $\lambda=\sqrt{5}/2$
$x=\frac{1}{\lambda}=2/\sqrt{5}$
$y=\frac{2}{\lambda}=4/\sqrt{5}$

when $\lambda=-\sqrt{5}/2$
$x=\frac{1}{\lambda}=-2/\sqrt{5}$
$y=\frac{2}{\lambda}=-4/\sqrt{5}$

$h(\frac{2}{\sqrt{5}},\frac{4}{\sqrt{5}})=4\sqrt{5}+20$
$h(-\frac{2}{\sqrt{5}},-\frac{4}{\sqrt{5}})=-4\sqrt{5}+20$

max height $=4\sqrt{5}+20$, minimum height $=-4\sqrt{5}+20$

(b) What if the spider only crawled on the upper half of the semicircle creating endpoints?

only one critical point survives $(2\sqrt{5}/5, 4\sqrt{5}/5)$

we must also test the endpoints i.e. when $y=0$

$x^2+y^2=4$
$x^2+(0)^2=4$
$x=\pm 2$

$h(2,0)=16$
$h(-2,0)=24$

maximum height $= 4\sqrt{5}+20$ minimum height 16